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Fuzzy Soft Partitions

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Abstract

In this paper, we generalized the concept of fuzzy partitions attributed to Dumitrescu [7] to fuzzy soft partitions based on fuzzy soft equality. Then, some properties relative to fuzzy soft sets operations will be studied. Finally, we introduce some main fuzzy soft partitions of a fuzzy soft set.

1 Introduction

Uncertainty data specially in complicated problems in economics, engineering, medical sciences and etc., ensure scientists and mathematicians to think about the multiple valued logics. Nowadays, there are several well-known theories to describe uncertainty, for instance, fuzzy set theory, rough set theory and other mathematical tools. But, all these theories have their inherited difficulties as pointed out by Molodtsove [11]. The theory of Soft sets which were introduced by Molodtsove in 1999 is one of the new mathematical tools for dealing with uncertainties. This theory is free of the difficulties affecting the existing methods. As a combination of fuzzy sets and soft sets. P.K. Maji and et.al. [9] initiated the concept of fuzzy soft sets. Since then, the appealing concepts of soft relations and fuzzy soft relations by M.I. Ali and M.J. Borah, et.al. respectively in [1, 5].

One of the most importance concepts in set theory is the concept of set partitions which has many applications in all branches of mathematics and other sciences. In fuzzy set theory we have various definitions for this concept, many of them based on t-norms and t-conorms. One can see some of them in [3]. One of celebrated definitions which has many applications is attributed to D. Dumitrescu [7] applied by herself to extend the concepts of fuzzy ergodic theory and by other authors such as R. Meisar and J. Rybarik for generalizing Entropy of fuzzy partitions [10], H.F. Pop and et.al. to fuzzy classification of the chemical elements [12] and A. Amo and et.al [2] to fuzzy classification systems and so on.

Therefore, with the increasing applications of soft set theory, generalizing the concept of "Partition" to this set theory will be phenomenal and useful.

2 Preliminaries

Let X be a universe. By a fuzzy set f on X, we mean a function $f : X \to [0,1] = I$. *Fuzzy null* set 0 on X defined by $0(x) = 0$ for each $x \in X$. Also, if E is a set of parameters on the elements of X and A is a nonempty subset of E, then a *soft set* F with respect to A denoted by (F, A) where $F : E \to 2^X$ is a function such that $F(x) = \emptyset$ for all $x \in E - A$. The collection of all fuzzy sets on X denoted by I^X. Set operations on fuzzy sets defines by t-norms and their duals t-conorms. According to [6] if T is a t-norm and S be it's dual then for fuzzy sets $f, g \in I^X$:

$$(f \cup g)(x) := T(f(x), g(x))$$
$$(f \cap g)(x) := S(f(x), g(x))$$

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f and g are disjoint fuzzy sets iff $f \cap g = 0$. If the chosen connectives are $T = T_\infty$ and $S = S_\infty$ the definitions above become

$$(f \cap g)(x) = T_\infty(f(x), g(x)) = max(f(x) + g(x) - 1, 0) \quad \forall x \in X$$
$$(f \cup g)(x) = S_\infty(f(x), g(x)) = min(f(x) + g(x), 1) \qquad \forall x \in X.$$

For every fuzzy set $f \in I^X$ it's complement is a fuzzy set f^c where $f^c(x) = 1 - f(x)$ for each $x \in X$.

According to [9], for any $A \subseteq E$, a pair (f, A) is called a fuzzy soft set over (or "on") (X, E), where f is a mapping from A into I^X, that is, for each $a \in A$, $f(a) = f_a : X \to I$ is a fuzzy set on X.

Definition 2.1. *The (restricted) intersection (union) of two fuzzy soft sets $(f, A), (g, B)$ over a common universe X, denoted by $(f, A) \cap (g, B)$ ($(f, A) \cup (g, B)$) is fuzzy soft set $(f \cap g, A \cap B)$ ($(f \cup g, A \cap B)$) where*

$$f \cap g : A \cap B \to I^X$$
$$(f \cup g : A \cap B \to I^X)$$

defines as $(f \cap g)_a = f_a \cap g_a$, ($(f \cup g)_a = f_a \cup g_a$) for all $a \in A \cap B$.

Null fuzzy soft set , denoted by $(\varnothing, E)$, if $\varnothing_a = 0$ for all $a \in E$. Whole fuzzy soft set, denoted by (X, E), if $X_a = 1$ for all $a \in E$.

Now, we generalize the concept of soft equality [13] to fuzzy soft sets by next definition.

Definition 2.2. *Let $(f, A), (g, B)$ be two fuzzy soft sets over the common universe (X, E). (f, A) is called soft equal to (g, B), denoted by $(f, A) =_s (g, B)$, if for all $a \in A \cup B$:*

$$\begin{cases} f_a = 0 & a \in A - B \\ f_a = g_a & a \in A \cap B \\ g_a = 0 & a \in B - A \end{cases}$$

As a dual definition we can do the next.

Definition 2.3. *Let $(f, A), (g, B)$ be two fuzzy soft sets over the common universe (X, E). (f, A) is called Soft equal to (g, B), denoted by $(f, A) =^s (g, B)$, if for all $a \in A \cup B$:*

$$\begin{cases} f_a = 1 & a \in A - B \\ f_a = g_a & a \in A \cap B \\ g_a = 1 & a \in B - A \end{cases}$$

Based on [13], one can easily state the next theorem.

Theorem 2.4. *For every fuzzy soft sets $(f, A), (g, B)$:*

$$(f, A) =^s (g, B) \iff (f, A)^c =_s (g, B)^c$$

where $(f, A)^c = (f^c, A)$.

According to above theorem, we use $=_s$, same results obtains when we replace $=_s$ by $=^s$.

Proposition 2.5. *For all $A \subseteq E$:*

1. $(0, E) =_s (0, A)$.

2. $(1, E) =^s (1, A)$

Proof. It's clear. □

Definition 2.6. *Product and difference of fuzzy soft sets $(f, A), (g, B)$ defined by:*

$$(f, A)(g, B) = (f.g, A \cap B) \quad \text{where for all } \; a \in A \cap B : (f.g)_a := f_a g_a$$

$$(f, A) - (g, B) = (f - g, A \cap B) \qquad \text{where for all } a \in A \cap B : (f - g)_a = max(f_a - g_a, 0).$$

Proposition 2.7. *For every fuzzy soft sets (f, A), (g, B) on (X, E), if $(f, A) = (g, B)$ then $(f, A) =_s (g, B)$.*

Proof. It is clear. ☐

The converse may be not true. For instance let $A = [0, \frac{2}{3}], B = [\frac{1}{3}, 1]$. Define:

$$f(a) = \begin{cases} 0 & , \quad 0 \leq a < \frac{1}{3} \\ a & , \quad \frac{1}{3} \leq a < \frac{2}{3} \end{cases} \quad , \quad g(b) = \begin{cases} b & , \quad \frac{1}{3} \leq b < \frac{2}{3} \\ 0 & , \quad \frac{2}{3} \leq b < 1 \end{cases}$$

It is easy to see that $(f, A) =_s (g, B)$ but $(f, A) \neq (g, B)$.

Proposition 2.8. *Let $(f, A), (g, B)$ and (h, C) are fuzzy soft sets over (X, E). Then*

1. $(f, A)(g, B) = (g, B)(f, A)$.
2. *if* $(f, A) =_s (g, B)$ *then* $(h, C)(f, A) =_s (h, C)(g, B)$.
3. $(f, A)(0, B) =_s (0, B)(f, A) =_s (0, A \cap B)$.
4. $(f, A)(1, B) =_s (1, B)(f, A) =_s (f, A \cap B)$.

Proof. 1. Obvious.
2. By definition, $(f, A) =_s (g, B)$ implies that

$$\begin{cases} f(a) = g(a) & a \in A \cap B \\ f(a) = 0 & a \in A - B \\ g(b) = 0 & b \in B - A \end{cases}$$

consequently

$$\begin{cases} h(a)f(a) = h(a)g(a) & a \in A \cap B \cap C = (A \cap C) \cap (B \cap C) \\ h(a)f(a) = 0 & a \in (A - B) \cap C = (A \cap C) - (B \cap C) \\ h(a)g(a) = 0 & a \in (B - A) \cap C = (B \cap C) - (A \cap C) \end{cases}$$

Thus, $(h, C)(f, A) =_s (h, C)(g, B)$.
3. By Proposition 2.7 it comes straightforward.
4. It is clear. ☐

Note that the converse of part 2 may be not hold i.e. "Elimination" property dose nor exists.

Example 2.9. *Let $A = \{1, 2, 3, 4, 6\}, B = \{2, 3, 4, 5, 7\}$ and $C = \{1, 2, 3, 5\}$ and $(h, C) = (0.5, C)$. Assume that*

$$f(a) = \begin{cases} 0.5 & , \quad a = 2, 3 \\ 0.7 & , \quad a = 6 \\ 0 & , \quad a = 1, 4 \end{cases} \quad \text{and} \quad g(b) = \begin{cases} 0.5 & , \quad b = 2, 3 \\ 0.2 & , \quad b = 7 \\ 0 & , \quad b = 4, 5 \end{cases}$$

therefore

$$(hf)(a) = \begin{cases} 0.25 & , \quad a = 2, 3 \\ 0 & , \quad a = 1 \end{cases} \quad \text{and} \quad (hg)(b) = \begin{cases} 0.25 & , \quad b = 2, 3 \\ 0 & , \quad b = 5 \end{cases}$$

Now, we can see $(h, C)(f, A) =_s (h, C)(g, B)$ but $(f, A) \neq_s (g, B)$.

Proposition 2.10. *Let $(f, A), (g, B)$ are two fuzzy soft sets on (X, E). Then*

$$\forall a \in A \cap B: \qquad f_a \cup g_a = f_a + g_a - (f_a \cap g_a).$$

Proof. For each $x \in X$, we have

$$\begin{array}{rcl} f_a(x) + g_b(x) - (f_a \cap g_b)(x) & = & f_a(x) + g_b(x) - max(f_a(x) + g_b(x) - 1, 0) \\ & = & min(f_a(x) + g_b(x) - f_a(x) - g_b(x) + 1, f_a(x) + g_b(x)) \\ & = & min(1, f_a(x) + g_b(x)) \\ & = & (f_a \cup g_b)(x). \end{array}$$

□

Definition 2.11. *Let (ϕ, C) be a fuzzy soft set and $(f, A) \subseteq (\phi, C)$. The soft complement of (f, A) relative to (ϕ, C) is a fuzzy soft set (g, B) such that $(f, A) \cup (g, B) =_s (\phi, C)$ and $(f, A) \cap (g, B) =_s (\varnothing, C)$.*

Corollary 2.12. *If (g, B) is soft complement of (f, A) relative to (ϕ, C) then $g_a = \phi_a - f_a$ for all $a \in A \cup B$.*

Definition 2.13. *A family $\{(f^i, A^i)\}_{i=1}^n$ of fuzzy soft sets is called disjoint iff*

$$(\bigcup_{i \neq j} (f^i, A^i) \bigcap (f^j, A^j) =_s (\varnothing, \bigcup_{i=1}^n A^i).$$

3 Fuzzy soft Partitions

Now, we able to state the concept of soft partitions.

Definition 3.1. *Let (ϕ, C) be a fuzzy soft set on X. A finite family $P = \{(f^i, A^i)\}_{i=1}^n$ of disjoint fuzzy soft sets is a finite fuzzy soft partition of (ϕ, C) iff $\bigcup_{i=1}^n (f^i, A^i) =_s (\phi, C)$. Each (f^i, A^i) is a soft cell (or atom) of the fuzzy soft partition P.*

Theorem 3.2. *$P = \{(f^i, A^i)\}_{i=1}^n$ is a fuzzy soft partition of (ϕ, C) iff $\sum_{i=1}^n f_a^i = \phi_a$ for all $a \in \bigcap_{i=1}^n A^i \cap C$.*

Proof. Assume that $\{(f^i, A^i)\}_{i=1}^n$ be a fuzzy soft partition for (ϕ, C). So, $\bigcap_{i=1}^n (f^i, A^i) =_s (\varnothing, \bigcap_{i=1}^n A^i)$ and $\bigcup_{i=1}^n (f^i, A^i) =_s (\phi, C)$. First equality implies that $\bigcap_{i=1}^n f_a^i = 0$ for each $a \in \bigcap_{i=1}^n A^i$. By Dumitrescu theorem [7],

$$\sum_{i=1}^n f_a^i = \phi_a \qquad \forall a \in \bigcap_{i=1}^n A^i \cap C.$$

In the other hand, when for an $a \in \bigcap_{i=1}^n A^i \cap C : \sum_{i=1}^n f_a^i = \phi_a$, against by Dumitrescu theorem, $\{f_a^i\}_{i=1}^n$ is a fuzzy partition for ϕ_a. Its means is $\bigcap_{i=1}^n f_a^i = \varnothing$ and $\bigcup_{i=1}^n f_a^i = \phi_a$ for all $a \in \bigcap_{i=1}^n A^i \cap C$. Thus, $(\bigcup_{i=1}^n f^i, \bigcap_{i=1}^n A^i) =_s (\phi, C)$ and $(\bigcap_{i=1}^n f^i, \bigcap_{i=1}^n A^i) =_s (\varnothing, C)$, which means is

$$\bigcap_{i=1}^n (f^i, A^i) =_s (\varnothing, \bigcap_{i=1}^n A^i)$$

and

$$\bigcup_{i=1}^n (f^i, A^i) =_s (\phi, C).$$

That is $\{(f^i, A^i)\}_{i=1}^n$ is fuzzy soft partition of (ϕ, C).

□

The class of all fuzzy soft partitions of (ϕ, C) will denote by $FSP(\phi, C)$.

Example 3.3. *Suppose that $X = \{1, 2, 3, 4\}$, E is any set of parameters on X and let $C = \{a, b, c\}$ be a subset of E. Given a fuzzy soft set (ϕ, C) as*

(ϕ, C)	ϕ_a	ϕ_b	ϕ_c
1	0.5	0.8	0
2	0.6	0.1	0.5
3	0.8	0.7	0.9
4	0.4	0.6	0.3

Let $A^1 = \{a, b, d\}, A^2 = \{a, b, c\}$ *and* $A^3 = \{a, b, d\}$. *Consider*

(f^1, A^1)	f^1_a	f^1_b	f^1_d
1	0.1	0.5	0.7
2	0.2	0	0.5
3	0.4	0.1	0.6
4	0.1	0.2	1

(f^2, A^2)	f^2_a	f^2_b	f^2_c
1	0.1	0.2	0
2	0.2	0	0.3
3	0.2	0.3	0.5
4	0.2	0.2	0.1

(f^3, A^3)	f^3_a	f^3_b	f^3_d
1	0.3	0.1	1
2	0.2	0.1	1
3	0.2	0.3	0.9
4	0.1	0.2	0.3

It is easy to check that the family $\{(f^1, A^1), (f^2, A^2), (f^3, A^3)\}$ *is a fuzzy soft partition for* (ϕ, C).

Example 3.4. *For every fuzzy soft set* (ϕ, C) *on* (X, E)*, the family* $\{(\phi, C), (\varnothing, C)\}$ *is a fuzzy soft partition for* (ϕ, C) *called triviall fuzzy soft partition. Also, if* (f, A) *be a fuzzy soft subset of* (ϕ, C) *then* $\{(f, A), (\phi - f, A)\}$ *is a fuzzy soft partition of* (ϕ, C) *where* $(\phi - f)(a) = \phi(a) - f(a)$ *for all* $a \in A$ *is the complement of* f *relative to* ϕ.

Theorem 3.5. *Consider that* $(X, E), (Y, F)$ *be two universes and* $(\phi, \psi) : (X, E) \to (Y, F)$ *be a pair of functions where* $\phi : X \to Y$ *and* $\psi : E \to F$. *Let* $Q = \{(g^j, B^j)\}_{j=1}^n$ *be a fuzzy soft partition of* (Y, F). *Then, define* $A^j = \psi^{-1}(B^j)$ *and* $f^j : A^j \to I^X$ *such that* $f^j(a) = f^j_a = g^j_{\psi(a)}$. *Then the family* $\{(f^j, A^j)\}_{j=1}^n$ *is a fuzzy soft partition for* (X, E).

Proof. Q is a fuzzy soft partition for (Y, F). So, $\sum_{j=1}^n g^j_b = 1$ for all $b \in \bigcap_{j=1}^n B^j$. For each $a \in \bigcap_{j=1}^n \psi^{-1}(B^j) = \psi^{-1}(\bigcap_{j=1}^n B^j) : \sum_{j=1}^n f^j(a) = \sum_{j=1}^n g^j_{\psi(a)} = 1$

□

Assume that P and Q are two fuzzy soft partitions of a fuzzy soft set (ϕ, C). Q is said to be a refinemet of P (and we write $P \prec Q$) if each cell of P is soft equal to a union of some atoms of Q.

Let $P = \{(f^i, A^i)\}_{i=1}^n$ and $Q = \{(g^j, B^j)\}_{j=1}^m$ be two finite fuzzy soft partitions of a fuzzy soft set (ϕ, C). The Algebraic join of P and Q denoted by $P \bigtriangledown Q$ defined by

$$P \bigtriangledown Q = \{(f^i, A^i)(g^j, B^j) | i = 1 \cdots n, j = 1 \cdots m\}.$$

Proposition 3.6. *If* P, Q *are finite fuzzy soft partitions of* (X, E) *then* $P \bigtriangledown Q$ *is also a fuzzy soft partition of* X *and* $P \prec P \bigtriangledown Q$.

Proof. We may write

$$\sum_{i=1}^n \sum_{j=1}^m (f^i g^j)(a) = \sum_{i=1}^n f^i(a) \sum_{j=1}^m g^j(a) = \sum_{i=1}^n f^i(a) = 1 \qquad \forall a \in \bigcap_{i,j} (A^i \cap B^j).$$

Since $(g^1, B^1), \cdots (g^m, B^m)$ are disjoint, we have

$$(f^i, A^i) =_s (f^i, A^i)(X, \bigcap_{i,j} (A^i \cap B^j)) =_s (f^i, A^i)(\bigcup_{j=1}^m, \bigcap_{j=1}^m A^i \cap B^j) =_s \bigcup_{j=1}^m (f^i g^j, A^i \cap b^j)$$

where each $(f^i g^j, A^i \cap B^j)$ belongs to $P \bigtriangledown Q$. □

Proposition 3.7. *For every two fuzzy soft sets* $(\phi, C), (\psi, D)$ *on a universe* (X, E)*, the family* $\{(\phi - (\phi \cap \psi), C \cap D), (\phi \cap \psi, C \cap D), (\psi - (\phi \cap \psi), C \cap D)\}$ *is a fuzzy soft partition for* $(\phi, C) \cup (\psi, D)$*.*

Proof. For all $a \in C \cap D$ we have

$$\begin{aligned}
[\phi - (\phi \cap \psi]_a(x) + (\phi \cap \psi)_a(x) + [\psi - (\phi \cap \psi)]_a(x) &= max[\phi_a(x) - (\phi \cap \psi)_a(x), 0] + \\
&\quad (\phi \cap \psi)_a(x) + max[\psi_a(x) - (\psi_a(x) - (\phi \cap \psi)_a(x), 0] \\
&= \phi_a(x) - (\phi \cap \psi)_a(x) + (\phi \cap \psi)_a(x) + \\
&\quad \psi_a(x) - (\phi \cap \psi)_a(x) \\
&= \phi_a(x) + \psi_a(x) - (\phi \cap \psi)_a(x) \\
&= (\phi \cup \psi)_a(x)
\end{aligned}$$

for all $x \in X$.

□

4 Conclusion

The introduced concept of fuzzy soft partition may provide a useful tool to build a fuzzy soft Information Theory. Other possible applications should be in Decision Theory, Ergodic Theory, Pattern Classification and Topology.

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