Journal of
Mathematics and Applications ISSN: 2717-3801
MathApp.ir

Mathematics and Applications 1 (2021) 18-24

Minimally Generated Subshifts

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Received November 2020 Revised December 2020

Abstract

We introduce the notion of a minimal generator G for the coded system X; that is a generator for coded system X whenever $u \in G$, then $u \notin W(\overline{< G \setminus \{u\} >})$. Such an X is called *minimally generated system.* We aim to introduce a class of minimally generated systems generated by some certain synchronizing blocks. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system.* A version of a theorem of K. Thomsen (2006) in minimally generated systems has been given. The derived set has been characterized as well.

1 Introduction

One of the most studied dynamical systems is a subshift of finite type (SFT). An SFT is a system whose set of forbidden blocks is finite [5]. Equivalently, an SFT X is a subshift whose any block of length greater than a certain number M is synchronizing; that is, if m is *any* block with $|m| \geq M$ and if v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics*: factors of SFT's. Synchronized systems has attracted much attention [1].

A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system.* Generators are needed for effective symbolic representation. For ergodic systems there is always a countable generator. If the entropy is finite then there is a finite generator, and there is an optimal bound for its cardinality expressible in terms of entropy [4].

Thomsen in [7] proves that $\lim_{k\to\infty} h(X_k) = h_{syn}(X)$ where

(1.1) $$h_{syn}(X) := \limsup_{n} \frac{1}{n} \log \left(\text{cardinal} \left\{a \in W_n(X) : mam \in W(X)\right\}\right)$$

where m is an arbitrary synchronizing block in $W(X)$ and X_k's are SFT approaching X from inside. A version of this theorem for totally synchronizing generated system subshifts has been given in section (4).

Set

(1.2) $$\partial X := \{x \in R(X) : w \subseteq x \Rightarrow w \notin S(X)\}$$

called the *derived shift space* of X. Then, $\partial(X)$ plays an important role in the dynamics of the system. As an example, a result in [6] states that in synchronized systems

$$h(X) = \max\{h_{\text{syn}}(X), h(\partial(X))\}.$$

Note that ∂X is empty for an SFT and it is sofic whenever X is sofic [7, Theorem 6.6]. In fact, for specified systems (containing sofic shifts) $h(X) = h_{\text{syn}}(X)$ [3, Page 16]. The notion of $\partial(X)$ for totally synchronizing generated systems introduced in section (5).

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Keywords: Strong synchroizing, Minimal generator, Half synchronizing, Entropy

MSC(2010): 37B10; 37B40; 54H20

2 Background and definitions

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from [5] and [2] for the relevant concepts.

First we present some elementary concept from [5]. Let $\mathcal{A}$ be an alphabet, that is a non-empty finite set. The full shift $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j-i+1$ by $x_{[i,\,j]} = x_i x_{i+1} \dots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of *admissible blocks* or just blocks in X. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks. Let $W_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair $u, v \in W(X)$, there is an N such that for each $n \geq N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$. A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [5, Theorem 3.1.5], but the converse is not true.

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of π. By π_u we mean a path labeled u.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) :\ \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* of $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if and only if $X = X_{\mathcal{G}}$ for a finite graph G [5, Proposition 3.2.10]. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels.

In this part we collect some information from [2]. Let X be a subshift and $x \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ (resp. $x_- = (x_i)_{i\leq 0}$) is called right (resp. left) infinite X-ray. Let $X^+ = \{x_+ : x \in X\}$ and $X^- = \{x_- : x \in X\}$. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ and for $m \in W(X)$ its follower set is $w_+(m) = \{x_+ \in X^+ : mx_+ \in X^+\}$. Analogously, we define predecessor sets $w_-(x_+) = \{x_- \in X^- : x_- x_+ \in X\}$ and $w_-(m) = \{x_- \in X^- : x_- m \in X^-\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = w_+(x_-), I_2 = w_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$ which is called the *magic vertex* in the Krieger graph. An irreducible shift space X is *synchronized system* if it has an synchronizing block. If X is an synchronized system with synchronizing m, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *Ficher cover* of X.

3 Minimal generator

A shift space that is the closure of the set of sequences obtained by freely concatenating the blocks in a list of countable blocks, called the set of generators, is a *coded system* [5]. Equivalently, a coded system is a shift space that can be presented by an irreducible countable labeled graph.

We aim to introduce a class of coded systems generated by synchronizing blocks.

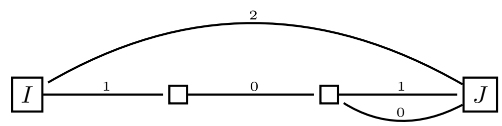

Figure 1: The grapg H for the cover of a synchronized system such that G_m is not a minimal generator for X_H.

Definition 3.1. *Let X be a coded system generated by G. Then, G is called* minimal *(resp.* weak minimal*), whenever $u \in G$, then $u \notin W(Z)$, (resp. $X \neq Z$) where $Z = \overline{< G \setminus \{u\} >}$. Such an X is called* minimally *(resp.* weak minimally*) generated system.*

Here, $< G >$ means the set of all concatenations of the elements of G. Clearly any minimally system is weak minimally. Let X be the Dyke system. Set

$$G_1 := \{(), (()), [()], ((())), [(())], ([()]), [[()]], \ldots\}$$

$$G_2 := \{[], ([]), [[]], [[[]]], ([[]]), [([])], (([])), \ldots\}.$$

Then, $G = G_1 \cup G_2$ is a weak minimal generator for the subshift $Z = \overline{< G >}$. But is not a minimal generator for it. Let $\emptyset \neq S \subseteq \mathbb{N}$. Then, $G := \{10^n 1 : n \in S\}$ is a minimal generator for subshift $X := \overline{< G >}$.

Definition 3.2. *Let X be a synchronized system. Call a block m an* strong synchronizing *for X if whenever e, e' are finite paths in Fischer cover X_0^+ labeled m, then $e = e'$.*

An irreducible shift space with an strong synchronizing block is called *strong synchronized*. Any strong synchronized system is a synchronized since any strong synchronizing block is a synchronizing block.

Let X be an strong synchronized system and $S_t(X)$ (resp. $S(X)$) denote the set of all strong synchronizing (resp. synchronizing) blocks for X. Note that if $m \in S(X)$ and $m \subseteq u$, then $u \in S(X)$. But it is not true when $m \in S_t(X)$. This fact can be seen by the fact that in Figure 1, $2 \in S_t(X)$ but $012 \notin S_t(X)$.

Let G be a minimal generator for a subshift X. Set G_{ts} denote the set of all $v \in G$ such that for all $u \in G$, there are not non empty blocks a, b such that $vu = avb$ or $uv = avb$.

Let G be a minimal generator for a subshift X with $G = G_{ts}$. Then, G is called a *totally synchronizing generator*. Such an X is called *totally synchronizing generated system*.

Next example shows that there are non-sofic but synchronized systems. Let P be the set of all prime numbers. Set $G := \{10^n 1 : n \in P\}$ and $X := \overline{< G >}$. Then, X is a totally synchronizing generated system. But the shift space X has infinitely many follower sets and so by [5, Theorem 3.2.10], X is not a sofic. Let $G = \{u_1, u_2, \ldots\}$ be a minimal generator for a subshift X. We give another right resolving and follower separated cover for X, denoted by $\mathcal{H}_G$ which is not necessarily Fischer cover of X. To do so fix $\{a_1, a_2, \ldots\} \subseteq \mathbb{N}$. Let the loop graph $\mathcal{G}$ has one vertex I_0 and infinite self loops e_i labeled a_i at that vertex ($i \geq 1$). We construct a new graph from $\mathcal{G}$ denoted by $\mathcal{G}_{u_i \hookrightarrow a_i}$ by replacing u_i for a_i whenever there is a path in $\mathcal{G}$ labeled a_i for all $i \geq 1$. We can suppose that $\mathcal{G}_{u_i \hookrightarrow a_i}$ is right resolving. Now let $\mathcal{H}_G$ be the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$. Then, by [5, Lemma 3.3.8] $X = X_{\mathcal{G}_{u_i \hookrightarrow a_i}} = X_{\mathcal{H}_G}$ and $\mathcal{H}_G$ is right resolving and follower separated. For instance see next example.

1. Let $X := \overline{< G >}$ where G is a minimal generator for X in example 3. Figure 2 shows $\mathcal{H}_G$ for G.

2. Let X be an strong synchronized system, $m \in S_t(X)$ and

$$G_m := \{ma_1, ma_2, \ldots\}$$

   be a minimal generator for X where $\{a_1, a_2, \ldots\} \subseteq W(X)$. Then, $X_0^+ = X_{\mathcal{H}_G}$ where $\mathcal{H}_G$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$ and $\mathcal{G}_{u_i \hookrightarrow a_i}$ is as the Figure 4.

The following gives a sufficient condition on a minimal generator G that the cover $\mathcal{H}_G$ be Fischer cover. For this we first need define the *magic block* m for a right reasolving cover if there is one and only one vertex I such that $m \in F_-(I)$ where

$$F_-(I) = \{\mathcal{L}\text{-labels of all finite paths terminating at } I\}.$$

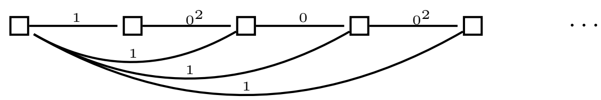

Figure 2: The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; $\mathcal{H}_G$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$ with $G := \{10^n 1 : n \in P\}$.

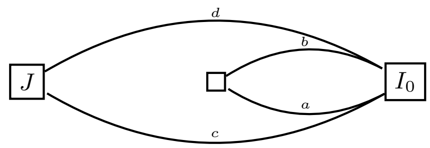

Figure 3: A subgraph of $\mathcal{H}_G$ where $ab = v_0 = bc$.

Proposition 3.3. *Let G be a minimal generator for the coded system X and assume that $v_0 \in G_{ts}$. Then,*

1. $\mathcal{H}_G = X_0^+$.
2. $G_{ts}(X) \subseteq S_t(X)$.

Proof. (i) To show that $\mathcal{H}_G = X_0^+$, it suffices by [2, Theorem 2.16] to show that $\mathcal{H}_G$ has a magic block. The construction of $\mathcal{H}_G$ shows that $v_0 \in F_-(I_0)$. Let $v_o \in F_-(J)$ and $J \neq I_0$. Then, there are non empty blocks a, b, c, d of X such that $ab = v_0 = bc$ and $v = cd$ as in Figure 3. Then, $v_0 v = a v_0 d$ and so $a = \varepsilon$ or $d = \varepsilon$ that are absurd and so v_0 is magic block for the $\mathcal{H}_G$ which set over claim.

(ii) Since there is exactly one path labeled v_0 in the Fischer cover $\mathcal{H}_G$, so v_0 is an strong synchronizing block of X. □

4 Computing Synchronized Entropy for a Totally Synchronizing Generated Systems

Let X be a shift space. The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |W_n(X)|$$

and for any synchronized system X, the *synchronized entropy* h_{syn} of X is defined by

$$h_{\text{syn}}(X) = \limsup_{n} \frac{1}{n} \log \left(\text{cardinal}\{a \in W_n(Y) : mam \in W(X)\} \right), \tag{4.1}$$

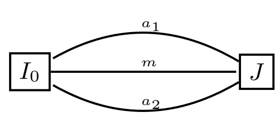

Figure 4: The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; $\mathcal{H}_G = (X_{\mathcal{H}_G})_0^+$ is the merged graph from $\mathcal{G}_{u_i \hookrightarrow a_i}$.

where $m \in W(X)$ is an arbitrary synchronizing block.

Lemma 4.1. *Let G be a minimal generator for the coded system X and $v \in G_{ts}$. Then,*

$$h_{syn}(X) = \limsup_n \frac{1}{n} \log |\{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

Proof. By Proposition 3.3, $v \in S_t(X)$ and so

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log |\{u \in W_n(X) :\ vuv \in W(X)\}|.$$

Since there is exactly one path labeled v in $\mathcal{H}_G = X_0^+$, so $vuv \in W(X)$ if and only if u is a finite concatenation of elements in G and so we are done. □

Let $X = \overline{\text{PerX}}$ be a shift space. For $s, t \in S(X)$ we write $s \sim t$ when there are blocks $u, v \in W(X)$ such that sut, $tvs \in W(X)$. Then, $\sim$ is an equivalence relation in $S(X)$. Consider an element $\alpha \in S(X)/\sim$. Let $X_{(\alpha,\,0)}$ denote the set of elements $x \in X$ for which

$$\sup\{\inf\{(j-i) \geq 0 :\ \exists w \in \alpha,\ w \subseteq x_{[i,\,j]}\}\}_{i \in \mathbb{Z}} \tag{4.2}$$

is finite. Here, we use the convention that $\inf \emptyset = \infty$. The subshift $X_{(\alpha,\,0)}$, will be called the *irreducible components* at level 0 in X [7].

Proposition 4.2. *Let G be a minimal generator generator for the coded system X and $v \in G_{ts}$. Then,*

$$h_{syn}(X) = \lim_{n\to\infty} \frac{1}{n gcd(X_{([v],\,0)})} \log |\{(v_1, \dots, v_k) \in G^k :\ \Sigma_{i=1}^k |v_i| = n gcd(\mathcal{L}(\mathcal{H}_G))\}|.$$

Proof. By Proposition 3.3, $v \in S_t(X)$ and so $v \in S(X)$. Hence X is a synchronized system. Thus by [7, Lemma 3.5], $\overline{X_{([v],\,0)}} = X$ and so by [7, Proposition 3.2],

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n\text{period}(X_{([v],\,0)})} \log |\{u \in W_{n\text{period}(X_{([v],\,0)})}(X) :\ vuv \in W(X)\}|. \tag{4.3}$$

But by Lemma 4.1,

$$\{u \in W_N(X) :\ vuv \in W(X)\} = \{(v_1, \dots, v_k) \in G^k :\ \Sigma_{i=1}^k |v_i| = N\}$$

where $N = n\text{period}(X_{([v],\,0)})$ and we are done. □

An implication of the above proposition is that if X is mixing synchronized, then by [7, Lemma 3.6], $\gcd(X_{([v],\,0)}) = 1$ and so

$$h_{\text{syn}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\{(v_1, \dots, v_k) \in G^k : \Sigma_{i=1}^k |v_i| = n\}|.$$

5 Derived Shift Space of a Totally Synchronizing Generated Systems

When X is a sofic shift with non-wandering part $R(X) := \overline{\text{PerX}}$, we can consider the shift space

$$\partial X = \{x \in R(X) :\ x \text{ consists of no blocks that are synchronizing for } R(X)\}$$

which is called the *derived shift space* of X. Since ∂X is a shift space we can continue, and consider $\partial(\partial X) = \partial^2 X$, $\partial(\partial^2 X) = \partial^3 X$, etc. We define the *depth* of X to be

$$Depth(X) = \sup\{n \in \mathbb{N} :\ \partial^n X \neq \emptyset\}.$$

The irreducible components at level 0 of $\partial^k X$ will be called the irreducible components of X at level k. They will be denoted by $X_{(\alpha,\,k)}$ [7].

Call an *internal subblock* of $u = a_1 a_2 \dots a_k$ to be a subblock of $a_2 \dots a_{k-1}$ which we denote it by u^0.

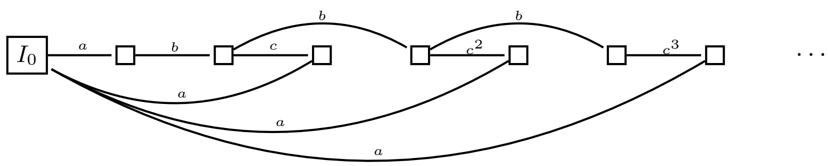

Figure 5: The graph $\mathcal{G}_{u_i \hookrightarrow a_i}$; merged graph $\mathcal{H}_G$ from $\mathcal{G}_{u_i \hookrightarrow a_i}$ have two irreducible components $\{b^\infty\}$ and $\{c^\infty\}$ at lavel 0 for $\partial(\overline{< G >})$.

Lemma 5.1. *Let X be a coded system generated by G. Then,*

$$\partial X = \{\cup_i u_i : \ u_i \subseteq v_{i_1} \dots v_{i_n} \ s.t. \ v_{i_j} \in G, u_i = (u_{i+1})^0 \ and \ u_i \notin S(X)\}.$$

Proof. Set $x := \cup_i u_i$ where $u_i \in W(X)$, $u_i = (u_{i+1})^0$ and $u_i \notin S(X)$. If there is a $m \in S(X)$ such that $m \subseteq x$, then $m \subseteq u_i$ for some $i \in \mathbb{N}$. That is absurd and so $x \in \partial X$.

Conversely, let $x \in \partial X$. Then, for each $i \in \mathbb{N}$, $x_{[-i,\,i]} \in W(X)$ is a block with no synchronizing subblock and a subblock of $v_{i_1} \dots v_{i_n}$ for some $v_{i_j} \in G$. Set $u_i := x_{[-i,\,i]}$ and then the conclusion is immediate. □

A *minimal synchronizing* block is a block whose proper subblocks are not synchronizing. Assume that all elements of the generator G of a coded system X are minimal synchronizing block. Then,

$$\partial X = \{\cup_n u_n : \ u_n \subsetneq v \text{ for some } v \in G \text{ and } u_n = (u_{n+1})^0\}.$$

Let X be a subshift over $\mathcal{A}$ generated by $G := \{u_i := ab^ic^ia : \ i \in \mathbb{N}\}$, where $a \in \mathcal{A}$, $\{b,\, c\} \subseteq W(X)$ and $a \notin bc$. Then,

1. $G = G_{ts}$.
2. $W(X) - S(X) = \{b^ic^j : \ i,\, j \geq 0\} \cup \{a\}$ and so $\partial X = \{b^\infty,\, c^\infty,\, b^\infty c^\infty\}$.

Example 5 gives a cover of an strong synchronized system of depth 2, that contains one irreducible component X at level 0 and two irreducible components $\{b^\infty\}$ and $\{c^\infty\}$ at level 1 (Figure 5). One can easily do this for infinitely many irreducible components. In fact let $P = \cup P_i$ be the set of all prime numbers such that $|P_i| = \infty$ and $P_i \cap P_j = \emptyset$ for all $i, j \in \mathbb{N}$. Set $G_i := \{2(10^{2i-1}1)^n 2 : \ n \in P_i\}$, where $i \in \mathbb{N}$ and Suppose that X be a subshift generated by $G := \cup_i G_i$. Then, it is not hard to see that $G = G_{ts}$ and $\partial X = \{(10^{2i-1}1)^\infty : \ i \geq 1\} \cup \{0^\infty\}$. Suppose that X is a subshift with a minimal generator G and $v \in G$. Then, $|V_v := \{I \in \mathcal{V}(\mathcal{H}_G) : \ t(\pi_v) = I\}| < \infty$.

If $w_+(I_0) = w_+(I)$ for all $I \in V_v$, then $v \in G_{ts}$ and so is an strong synchronizing block.

Thanks for your attention.

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