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On the spectral property of local shift-splitting preconditioner for double saddle point problems

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Abstract

This paper proposes a local shift-splitting preconditioner for the double saddle point matrices. Some properties of the local shift-splitting preconditioned double saddle point matrix are studied. Finally, numerical experiments of a model Stokes problem are presented to show the effectiveness of the proposed preconditioner.

1 Introduction

We consider the solution of the system of linear equations with the following block structure

$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T & C^T \\ -B & 0 & 0 \\ -C & 0 & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ are symmetric positive definite(SPD), $B \in \mathbb{R}^{m\times n}$ with rank$(B) = m < n$, $C \in \mathbb{R}^{p\times n}$, $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. Throughout the paper we assume that matrices A, B, C and D are large and sparse. Also, we suppose that $n \geq m + p$.

The linear system (1.1) is called double saddle point problem. Systems of the form (1.1) arise in a variety of scientific and engineering applications, including computational fluid dynamics [8], mixed finite element approximation and mixed-hybrid formulations of second-order elliptic PDEs[6], optimization [9], optimal control [10], weighted and equality constructed least squares estimation [11], structural analysis[12], electrical networks [12], inversion of geophysical data [13], computer graphics [14], and so on.

The solvability conditions for (1.1) and the uniqueness of its solution have been investigated in [2] which are summarized in the following proposition.

Proposition 1.1. *Assume that A and D are symmetric positive definite(SPD). Then, matrix $\mathcal{A}$ is invertible if and only if B has full row rank.*

Also, the following proposition gives a necessary and sufficient conditions for the invertibility of the coefficient matrix $\mathcal{A}$ in (1.1).

Proposition 1.2. *Let A be SPD and assume that B and C have full row rank. Consider the linear system* (1.1) *with* $D = 0$. *Then,* $range(B^T) \cap range(C^T) = \{0\}$ *is a necessary and sufficient condition for the coefficient matrix $\mathcal{A}$ to be invertible.*

Since the block matrix $\mathcal{A}$ in (1.1) is large and sparse, iterative methods are suitable for solving double saddle point problem (1.1). Some efficient iterative methods for saddle point problems have been studied in many literatures, including Uzawa-type methods [2], Hermitian and skew-Hermitian splitting(HSS) iterative methods and its variant schemes [15], preconditioned Krylov subspace iterative methods [1]. Benzi et.al. [1] reviewed various of preconditions and iterative solvers for solving saddle point problems.

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Both Uzawa-type stationary methods and block preconditioned Krylov subspace methods are discussed in [2] for double saddle point problem (1.1).

In this paper, we establish local shift-splitting(LSS) method and local shift-splitting precondition for solving the large sparse nonsingular saddle point problem (1.1). The LSS method is based on the shift splitting of the coefficient matrix $\mathcal{A}$ in (1.1) with a parameter α for local shift splitting approach. Also we introduce LSS precondition for Krylove subspace method, like GMRES.

2 Main results

As is well known, stationary iterative schemes for solving $\mathcal{A}u = b$ are uniquely associated with a given splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ where $\mathcal{M}$ is nonsingular. More precisely, an iterative method produces a sequence of approximate solutions as follows

$$u_{k+1} = \mathcal{T}u_k + \mathcal{M}^{-1}b, \quad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $\mathcal{T} = \mathcal{M}^{-1}\mathcal{N}$ is iteration matrix and the method start with initial guess u_0. It is proved that (2.1) is convergent for any initial guess if and only if $\rho(\mathcal{T}) < 1$, where $\rho(A)$ stands for spectral radius of matrix A.

Yin and Su in [] proposed a shift-splitting precondition

$$\mathcal{M} = \frac{1}{2}(\alpha I + \mathcal{A})$$

where α is a positive constant. In this paper, we modify this idea and propose a local shift splitting precondition and some useful properties of the local shift splitting preconditioned matrix are studied.

Based on the iteration methods studied in [3, 4], a local shift-splitting of the double saddle point matrix $\mathcal{A}$ can be constructed as follows

$$\begin{aligned} \mathcal{A} &= \mathcal{P}_{LSS} - \mathcal{Q}_{LSS} \\ &= \frac{1}{2}\begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -A & -B^T & -C^T \\ B & \alpha I & 0 \\ C & 0 & -D \end{bmatrix}, \end{aligned} \tag{2.2}$$

where $\alpha > 0$ is a constant and I is the identity matrix with appropriate dimension. By this special splitting we imply the following local shift splitting iteration method can be defined for solving the double saddle point problem (1.1).

The local shift-splitting iteration method Given an initial guess $u^{(0)}$, for $k = 0, 1, 2, \ldots$, until $\{u^{(k)}\}$ converges, compute

$$\frac{1}{2}\begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix} u^{(k+1)} = \frac{1}{2}\begin{bmatrix} -A & -B^T & -C^T \\ B & \alpha I & 0 \\ C & 0 & -D \end{bmatrix} u^{(k)} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2.3}$$

where α is a given positive constant.

It is important to be noted that any matrix splitting not only can lead to a splitting iteration method, but also can naturally induce splitting precondition for the Krylov subspace methods, like GMRES. The splitting precondition corresponds to the local shift- splitting iteration (3.7) is given by

$$\mathcal{P}_{LSS} = \frac{1}{2}\begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix}, \tag{2.4}$$

we call this precondition the local shift-splitting precondition for the saddle point matrix $\mathcal{A}$ in (1.1). We note here that at each step of the shift-splitting iteration or applying the shift-splitting precondition $\mathcal{P}_{LSS}$ within a Krylov subspace method, we need to solve a linear system with $\mathcal{P}_{LSS}$ as the coefficient matrix, as follows

$$\frac{1}{2}\begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix} z = r,$$

for a given vector r at each step. We can do the following matrix factorization for $\mathcal{P}_{LSS}$

$$\mathcal{P}_{LSS} = \frac{1}{2}\begin{bmatrix} I & \frac{1}{\alpha}B^T & C^TD^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} S & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & D \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ -\frac{1}{\alpha}B & I & 0 \\ -D^{-1}C & 0 & I \end{bmatrix}, \tag{2.5}$$

where $S = A + \frac{1}{\alpha}B^TB + C^TD^{-1}C$.

Let $r = [r_1^T, r_2^T, r_3^T]^T$ and $z = [z_1^T, z_2^T, z_3^T]^T$, where $r_1, z_1 \in \mathbb{R}^n$, $r_2, z_2 \in \mathbb{R}^m$ and $r_3, z_3 \in \mathbb{R}^p$. we have

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 2\begin{bmatrix} I & 0 & 0 \\ \frac{1}{\alpha}B & I & 0 \\ D^{-1}C & 0 & I \end{bmatrix}\begin{bmatrix} S & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & D \end{bmatrix}^{-1}\begin{bmatrix} I & -\frac{1}{\alpha}B^T & -C^TD^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}. \tag{2.6}$$

Hence, we can derive the following algorithm in which for a given $r = [r_1^T, r_2^T, r_3^T]^T$, one can compute the vector $z = [z_1^T, z_2^T, z_3^T]^T$ by (2.6).

Algorithm 1. *Computing* $z=\mathcal{P}_{LSS}^{-1}r$

(1) *Solve* $Dw = 2r_3$

(2) $w_1 = 2\left(r_1 - \frac{1}{\alpha}B^Tr_2\right) - C^Tw$

(3) *Solve* $(A + \frac{1}{\alpha}B^TB + C^TD^{-1}C)z_1 = w_1$

(4) $z_2 = \frac{1}{\alpha}(Bz_1 + 2r_2)$

(5) *Solve* $Dv = Cz_1$

(6) $z_3 = v + w$

From Algorithm 1, we can see that at each iteration, it is required to solve linear systems with the coefficient matrices D and $A + \frac{1}{\alpha}B^TB + C^TD^{-1}C$. Fortunately, the matrices D and $A + \frac{1}{\alpha}B^TB + C^TD^{-1}C$ are symmetric positive definite for all $\alpha > 0$. Therefore, in practical implementations, we can employ the conjugate gradient (CG) or the preconditioned conjugate gradient (PCG) method by a prescribed accuracy. Besides, the sub-linear systems can be solved by some direct methods, such as the Cholesky or LU factorization in combination with approximate minimum degree(AMD) permutation for better reordering of the results.

3 Convergence analysis

Now, we turn to study the convergence of the local shift-splitting iteration method. Note that the iteration matrix of the local shift-splitting iteration is

$$\mathcal{T}_{LSS} = \begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix}^{-1}\begin{bmatrix} A & -B^T & -C^T \\ B & \alpha I & 0 \\ C & 0 & D \end{bmatrix} \tag{3.1}$$

As it is well-known that the necessary and sufficient condition for convergence of the local shift-splitting iteration is $\rho(\mathcal{T}_{LSS}) < 1$.

Let λ be an eigenvalue of $\mathcal{T}_{LSS}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the corresponding eigenvector. Then we have

$$\begin{bmatrix} A & -B^T & -C^T \\ B & \alpha I & 0 \\ C & 0 & D \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & D \end{bmatrix}\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

or equivalently,

$$(\lambda+1)Ax + (\lambda+1)B^T y + (\lambda+1)C^T z = 0 \tag{3.2}$$
$$(\lambda+1)Bx + (1-\lambda)\alpha y = 0 \tag{3.3}$$
$$(\lambda+1)Cx - (1+\lambda)Dz = 0 \tag{3.4}$$

To study the convergence of the shift-splitting iteration, we first give the following lemma.

Lemma 3.1. *Let A and D be symmetric positive definite matrices, and B have full row rank. Let $\mathcal{T}_{LSS}$ be defined as in (3.1). If λ is an eigenvalue of $\mathcal{T}_{LSS}$, then $\lambda \neq \pm 1$.*

Proof. If $\lambda = 1$, from (3.2)-(3.4) we have

$$\begin{cases} Ax + B^T y + C^T z = 0 \\ -Bx = 0 \\ -Cx + Dz = 0 \end{cases} \tag{3.5}$$

By Proposition 1.1, one can get the coefficient matrix of (3.5) in nonsingular. Hence $x = 0, y = 0$ and $z = 0$, which is contradiction with the assumption that $(x; y; z)$ is an eigenvector of the iteration matrix $\mathcal{T}_{LSS}$. So $\lambda \neq 1$. Now, if $\lambda = -1$, then from (3.2)-(3.4) we get

$$\alpha y = 0,$$

Since $\alpha > 0$, we obtain $y = 0$. On the other hand, for any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ the triple $(x; 0; z)$ is an eigenvector corresponding to $\lambda = -1$. In particular we have $(0; 0; 0)$ as an eigenvector of $\lambda = -1$, which also contradicts the assumption that $(x; y; z)$ is nonzero vector. Therefore $\lambda \neq -1$. □

Similar to Theorem 2.1 in [4], next Theorem could be proved for the convergence of the local shift splitting scheme proposed in this paper.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ be symmetric positive definite matrices and B has full row rank, and let α be a positive number. Then, we have*

$$\rho(\mathcal{T}_{LSS}) < 1, \quad \forall \alpha > 0.$$

Proof. If $x = 0$ and $z = 0$, from (3.2)-(3.4), we get $y = 0$, which contradicts the fact that $(x^T, y^T, z^T)^T$ is an eigenvector of the iteration matrix $\mathcal{T}_{LSS}$. Without loss of generality, we assume $x \neq 0$ and $\|x\|_2 = 1$. Multiplying left sides of (3.2) by x^* yields

$$(\lambda+1)x^* Ax + (\lambda+1)(Bx)^* y + (\lambda+1)(Cx)^* z = 0$$

Due to the fact that $\lambda \neq -1$, we can drive from (3.3) and (3.4)

$$Bx = \frac{(\lambda-1)\alpha}{\lambda+1} y, \quad Cx = Dz \tag{3.6}$$

Substituting (3.6) in (3.2) yields

$$x^* Ax + \alpha\overline{\mathbf{w}}(y^* y) - z^* Dz = 0$$

where $\mathbf{w} = \dfrac{1-\lambda}{1+\lambda}$. It is obvious that $\mathbf{w} \neq 0$. So, we have

$$\mathbf{w}\alpha + \alpha\overline{\mathbf{w}}(y^* y) = z^* Dz + x^* Ax,$$

which implies

$$\Re(\mathbf{w}) = \frac{z^* Dz + x^* Ax}{\alpha(1 + y^* y)} > 0$$

and

$$|\lambda| = \frac{|1-\mathbf{w}|}{|1+\mathbf{w}|} = \sqrt{\frac{(1-\Re(\mathbf{w}))^2 + \Im((\mathbf{w})^2}{(1+\Re(\mathbf{w}))^2 + \Im((\mathbf{w})^2}} < 1$$

There is no case that $x = 0$, and $z \neq 0$. The proof is complete. □

We propose using the Krylov subspace method like GMRES, or its restarted version GMRES(m) to accelerate the convergence of the iteration. It is easy to see that the linear system $\mathcal{A}u = b$ is equivalent to the linear system [5]

$$(I - \mathcal{T}_{LSS})u = \mathcal{P}_{LSS}^{-1}\mathcal{A}u = \mathcal{P}_{LSS}^{-1}b.$$

Theorem **??** shows that this iteration is unconditionally convergent. This property is critical, since it implies that the spectrum of the preconditioned matrix lies entirely in a circle centered at $(1, 0)$ with radius 1, which is a desirable property for Krylov subspace acceleration. In the following Theorem we study spectral properties of the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$.

Theorem 3.3. *Assume that* $A \in \mathbb{R}^{n\times n}$ *and* $D \in \mathbb{R}^{p\times p}$ *are symmetric positive definite(SPD),* $B \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank}(B) = m < n$, $C \in \mathbb{R}^{p\times n}$. *Let* α *be a positive constant. The local shift-splitting precondition* $\mathcal{P}_{LSS}$ *defined as in* (2.2). *Then the preconditioned matrix* $\mathcal{P}_{LSS}$ *has an eigenvalue 2 with multiplicity* $n + p$, *the remaining eigenvalues are* $\frac{2\sigma_i^2}{\alpha+\sigma_i^2}$ *for* $i = 1, 2, \ldots, m$, *where* σ_i *is the positive singular values of the matrix* $BA^{-\frac{1}{2}}$.

Proof. By using the factorization in (3.7), we have

$$\mathcal{P}_{LSS}^{-1}\mathcal{A} = \begin{bmatrix} 2I & 2S^{-1}B^T & 0 \\ 0 & \frac{2}{\alpha}BS^{-1}B^T & 0 \\ 0 & 2D^{-1}CS^{-1}B^T & 2I \end{bmatrix}. \tag{3.7}$$

Consider the block diagonal matrix

$$P = \begin{bmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & D^{\frac{1}{2}} \end{bmatrix}.$$

Then the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ is similar to the following matrix $\tilde{P}$

$$\tilde{P} = P\mathcal{P}_{LSS}^{-1}\mathcal{A}P^{-1} = \begin{bmatrix} 2I & 2S_1^{-1}A^{-\frac{1}{2}}B^T & 0 \\ 0 & \frac{2}{\alpha}B^{-\frac{1}{2}}S_1^{-1}A^{-\frac{1}{2}}B^T & 0 \\ 0 & 2D^{-\frac{1}{2}}CA^{-\frac{1}{2}}S_2^{-1}A^{-\frac{1}{2}}B^T & 2I \end{bmatrix},$$

where $S_1 = I + \frac{1}{\alpha}A^{-\frac{1}{2}}B^TBA^{-\frac{1}{2}}$ and $S_2 = I + \frac{1}{\alpha}A^{-\frac{1}{2}}B^TBA^{-\frac{1}{2}} + A^{-\frac{1}{2}}C^TD^{-\frac{1}{2}}D^{-\frac{1}{2}}CA^{-\frac{1}{2}}$. Let σ_i for $i = 1, 2, \ldots, m$ and δ_j for $j = 1, 2, \ldots, p$ are the positive singular values of the matrices $BA^{-\frac{1}{2}}$ and $D^{-\frac{1}{2}}CA^{-\frac{1}{2}}$, respectively. Consider the singular value decomposition of the matrix$BA^{-\frac{1}{2}} = U_1^T\tilde{\Sigma}_1V_1 = U_1^T[\Sigma_1 \quad 0]V_1$ and $D^{-\frac{1}{2}}CA^{-\frac{1}{2}} = U_2^T\tilde{\Sigma}_2V_2 = U_2^T[\Sigma_2 \quad 0]V_2$, where $U_1 \in \mathbb{R}^{m\times m}$, $V_1 \in \mathbb{R}^{n\times n}$, $U_2 \in \mathbb{R}^{p\times p}$ and $V_2 \in \mathbb{R}^{n\times n}$ are orthogonal matrices. Also $\Sigma_1 = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ and $\Sigma_2 = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_p)$ are diagonal matrices with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$ and $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_p > 0$ being the nonzero singular values of $BA^{-\frac{1}{2}}$ and $D^{-\frac{1}{2}}CA^{-\frac{1}{2}}$, respectively.

Then $\tilde{P}$ can be written as

$$\begin{bmatrix} 2I & 2(I + \frac{1}{\alpha}V_1^T\tilde{\Sigma}_1^T\tilde{\Sigma}_1V_1)^{-1}V_1^T\tilde{\Sigma}_1^TU_1 & 0 \\ 0 & \frac{2}{\alpha}U_1^T\tilde{\Sigma}_1V_1(I + \frac{1}{\alpha}V_1^T\tilde{\Sigma}_1^T\tilde{\Sigma}_1V_1)^{-1}V_1^T\tilde{\Sigma}_1^TU_1 & 0 \\ 0 & 2U_2^T\tilde{\Sigma}_2V_2(I + \frac{1}{\alpha}V_1^T\tilde{\Sigma}_1^T\tilde{\Sigma}_1V_1 + V_2^T\tilde{\Sigma}_2^T\tilde{\Sigma}_2V_2)^{-1}V_1^T\tilde{\Sigma}_1^TU_1 & 2I \end{bmatrix}.$$

Define

$$Q = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & U_2 \end{bmatrix}.$$

Then Q is an orthogonal matrix and it holds that

$$\hat{P} = Q\tilde{P}Q^T = \begin{bmatrix} 2I & 0 & 2(I + \frac{1}{\alpha}\Sigma_1^2)^{-1}\Sigma_1 & 0 \\ 0 & 2I & 0 & 0 \\ 0 & 0 & \frac{2}{\alpha}\Sigma_1(I + \frac{1}{\alpha}\Sigma_1^2)^{-1}\Sigma_1 & 0 \\ 0 & 0 & 2\tilde{\Sigma}_2(V_1V_2^T + \frac{1}{\alpha}V_1\tilde{\Sigma}_1^T\tilde{\Sigma}_1V_2^T + V_1V_2^T\tilde{\Sigma}_2^T\tilde{\Sigma}_2)^{-1}\tilde{\Sigma}_1 & 2I \end{bmatrix}.$$

It is easy to check that the matrix $\hat{P}$ has an eigenvalue 2 with multiplicity $n + p$, the remaining eigenvalues are of the form $\frac{2\sigma_i^2}{\alpha+\sigma_i^2}$ for $i = 1, 2, \ldots, m$. □

Theorem 3.4. *Let the local shift-splitting precondition $\mathcal{P}_{LSS}$ be defined as in* (2.2), *then the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ is at most $m+1$. Thus, the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}_{LSS}^{-1}\mathcal{A}, b)$ is at most $m+1$.*

Proof. We know that the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ takes the form

$$\mathcal{P}_{LSS}^{-1}\mathcal{A} = \begin{bmatrix} 2I & \Theta_2 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & \Theta_3 & 2I \end{bmatrix}, \tag{3.8}$$

where $\Theta_1 = \frac{2}{\alpha}BS^{-1}B^T \in \mathbb{R}^{m\times m}$, $\Theta_2 = 2S^{-1}B^T \in \mathbb{R}^{m\times m}$ and $\Theta_3 = 2D^{-1}CS^{-1}B^T \in \mathbb{R}^{m\times m}$. Let μ_i for $i = 1, 2, \ldots, m$ be the eigenvalues of Θ_1. Note that μ_i for $i = 1, 2, \ldots, m$ are also the eigenvalue of the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$. Then from (3.8), we obtain the characteristic polynomial of the preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ is

$$\Phi(\lambda) = (\lambda - 2)^{n+p}\prod_{i=1}^{m}(\lambda - \mu_i).$$

Consider the polynomial $\Psi(\lambda) = (\lambda - 2)\prod_{i=1}^{m}(\lambda - \mu_i)$ of degree $m+1$, and expanding the polynomial $\Psi(\mathcal{P}_{LSS}^{-1}\mathcal{A})$, we obtain

$$(\mathcal{P}_{LSS}^{-1}\mathcal{A} - 2I)\prod_{i=1}^{m}((\mathcal{P}_{LSS}^{-1}\mathcal{A} - \mu_i I) = \begin{bmatrix} 0 & \Theta_2\prod_{i=1}^{m}(\Theta_1 - \mu_i I) & 0 \\ 0 & (\Theta_1 - 2I)\prod_{i=1}^{m}(\Theta_1 - \mu_i I) & 0 \\ 0 & \Theta_3\prod_{i=1}^{m}(\Theta_1 - \mu_i I) & 0 \end{bmatrix}.$$

Since μ_i for $i = 1, 2, \ldots, m$ are the eigenvalues of Θ_1, by Cayley-Hamilton theorem, we have

$$\prod_{i=1}^{m}(\Theta_1 - \mu_i I) = 0.$$

Therefor, from [5], we know that the degree of the minimal polynomial is equal to the dimension of the corresponding Krylov subspace $\mathcal{K}(\mathcal{P}_{LSS}^{-1}\mathcal{A}, b)$. This completes the proof. □

Remark 3.5. *In directly consequence of the eigenvalue distribution given in Theorem 3.4, we conclude that any Krylov subspace iterative method with an optimality or Galerkin property, such as GMRES, will terminate in at most $m+1$ iterations with the solution to a linear system of the form* (1.1) *if the local shift-splitting precondition $\mathcal{P}_{LSS}$ is used.*

4 Numerical experiments

In this section, we present some numerical experiments to illustrate the effectiveness of the local shift-splitting precondition for double saddle point problems arising from a model Stokes equation. In practical computations, we use left LSS preconditioning with restarted GMRES(♯) as the Krylov subspace method. Here, the integer ♯ in GMRES(♯) denotes that the algorithm is restarted after every ♯ iterations. In this paper, we take $\sharp = 30$. All runs are started from the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} \leq 10^{-6}$ where $r^{(k)} = b - \mathcal{A}u^{(k)}$ is the residual at the k-th iteration, or if the prescribed iteration number $k_{\max} = 5000$ is exceeded. The numerical results are compared with the preconditioned GMRES(♯) method by the well-known HSS(Hermitian and Skew-Hermitian Splitting)precondition and also GMRES without preconditioning. All runs are performed in MATLAB R2015a on an Intel Core i7 Laptop with 8G RAM.

Example 4.1. *By the finite difference scheme of the Stokes problem, the sub matrices of the coefficient matrix in the double saddle point problems have the following form*

$$A = \begin{bmatrix} I\otimes T + T\otimes I & 0 \\ 0 & I\otimes T + T\otimes I \end{bmatrix} \in \mathbb{R}^{2q^2\times 2q^2}, \qquad B^T = \begin{bmatrix} I\otimes F \\ F\otimes I \end{bmatrix} \in \mathbb{R}^{2q^2\times q^2},$$

$$D = I\otimes T + T\otimes I \in \mathbb{R}^{q^2\times 2q^2}, \qquad C^T = \begin{bmatrix} I\otimes F \\ F\otimes I \end{bmatrix} \in \mathbb{R}^{2q^2\times q^2},$$

where

$$T = \frac{\nu}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{q\times q}, \quad F = \frac{1}{h}\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{q\times q},$$

with $\otimes$ *being the Kronecker product symbol and* $h = \frac{1}{q+1}$ *the discretization mesh size.*

For this example, we have $n = 2q^2$, $m = q^2$ and $p = q^2$. Hence the total number of variables is $n + m + p = 4q^2$.

Table 1: Numerical results for solving Example 4.1 with $\nu = 0.1$

Grid		$\mathcal{I}$	$\mathcal{P}_{LSS}$	$\mathcal{P}_{HSS}$
8×8				
	IT	7(6)	1(2)	3(30)
	CPU	0.142	0.006	0.981
	ERR	9.7066e-07	5.8354e-07	6.3790e-07
16×16				
	IT	12(20)	1(2)	6(28)
	CPU	3.049	0.126	39.981
	ERR	9.6979e-07	4.4991e-07	9.1277e-07
24×24				
	IT	24(26)	1(2)	†
	CPU	35.542	0.886	†
	ERR	9.9043e-07	3.3027e-07	†

Table 2: Numerical results for solving Example 4.1 with $\nu = 0.01$

Grid		$\mathcal{I}$	$\mathcal{P}_{LSS}$	$\mathcal{P}_{HSS}$
8×8				
	IT	47(29)	1(2)	3(28)
	CPU	1.120	0.006	0.936
	ERR	9.9860e-07	5.1657e-08	7.6184e-07
16×16				
	IT	95(20)	1(2)	7(18)
	CPU	25.887	0.130	40.941
	ERR	9.9885e-07	6.5169e-08	9.1554e-07
24×24				
	IT	124(16)	1(2)	†
	CPU	182.438	0.888	†
	ERR	9.9933e-07	6.3544e-08	†

In Tables 1 and 2, we list the numerical results corresponding to the two ν, i.e. $\nu = 0.1, 0.01$. For each ν, three different q are used, i.e. $q = 8, 16, 24$. In these tables, $\mathcal{I}$, $\mathcal{P}_{LSS}$ and $\mathcal{P}_{HSS}$ denote the GMRES(30) method without preconditioning, with the left LSS preconditioning and with the left HSS preconditioning, respectively. The parameter α in the $\mathcal{P}_{LSS}$ is taken the same the viscosity ν. IT, CPU and ERR stand for the iteration numbers, the elapsed CPU times (in seconds) and the relative error, respectively. To demonstrate more efficient of shift-splitting method, the HSS precondition that is defined as follows:

$$\mathcal{P}_{HSS} = \begin{bmatrix} \alpha I + A & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & \alpha I + D \end{bmatrix} \begin{bmatrix} \alpha I & B^T & C^T \\ -B & \alpha I & 0 \\ -C & 0 & \alpha I \end{bmatrix},$$

is considered here, for which numerical results are obtained worse than GMRES without preconditioning in Tables 1 and 2.

In this example, we test two μ, i.e. $\nu = 0.1, 0.01$ for eigenvalue distribution of the saddle point matrix and the LSS preconditioned saddle point matrices. For each ν, two different q are used, i.e. $q = 32, 40$. Figure 1 and Figure 2 plots the eigenvalue distribution of the double saddle point matrix $\mathcal{A}$ for problems with 32×32 grids with $\nu = 0.1$ and

40×40 grids with $\nu = 0.01$, respectively. In Figure 1 and Figure 2, 'No preconditioning' and 'LSS' stand for the original double saddle point matrix $\mathcal{A}$ and the local shift-splitting preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$, respectively. Figure 1 and Figure 2 show that the eigenvalues of the LSS preconditioned matrix is more cluster than those of the double saddle point matrix $\mathcal{A}$. From Figure 1 and Figure 2, we see that most eigenvalues of the local shift-splitting preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$ are 2. These results are in good agreement with the theoretical ones in Theorem 3.3 and 3.4.

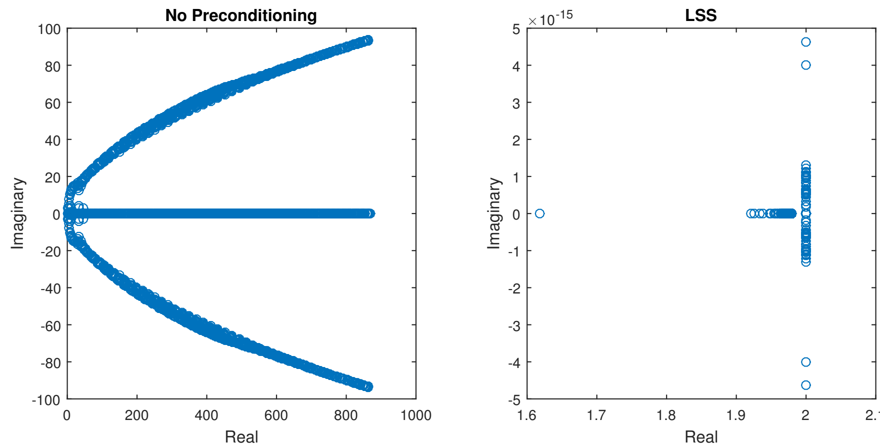

Figure 1: Eigenvalue distribution for Example 4.1 with 32×32 grids for $\nu = 0.1$

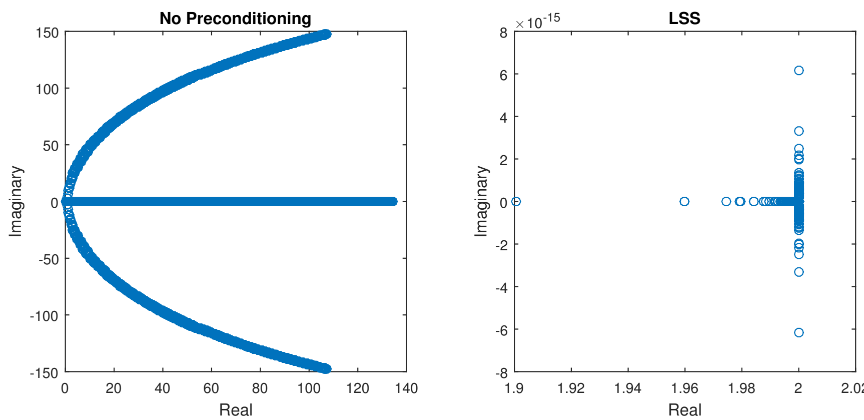

Figure 2: Eigenvalue distribution for Example 4.1 with 40×40 grids for $\nu = 0.01$

Example 4.2. *In the second test, we consider the saddle point* (1.1) *with the matrices* A, B, C *and* D *defined as follows*

$$A = (a_{ij}) = \begin{cases} i+1, & i = j \\ 1, & |i-j| = 1 \\ 0, & otherwise \end{cases} \in \mathbb{R}^{n\times n}, \quad D = (d_{ij}) = \begin{cases} i+1, & i = j \\ 1, & |i-j| = 1 \\ 0, & otherwise \end{cases} \in \mathbb{R}^{p\times p}$$

$$B = (b_{ij}) = \begin{cases} i, & j = i+n-m \\ 0, & otherwise \end{cases} \in \mathbb{R}^{m\times n}, \quad C = (c_{ij}) = \begin{cases} i, & j = i+n-p \\ 0, & otherwise \end{cases} \in \mathbb{R}^{p\times n}$$

This nonsingular double saddle point problem is induced from the literature [16].

In Tables 3, we list the numerical results corresponding to $\alpha = 0.01$ and different values of m, n and p. So, $\mathcal{I}$, $\mathcal{P}_{LSS}$ and $\mathcal{P}_{HSS}$ denote the GMRES(30) method without preconditioning, with the left LSS preconditioning and with the left

Table 3: Numerical results for solving Example 4.2 with $\alpha = 0.01$

Grid		$\mathcal{I}$	$\mathcal{P}_{LSS}$	$\mathcal{P}_{HSS}$
$n = 600$				
$m = 550$	IT	48(5)	1(3)	4(27)
$p = 50$	CPU	1.212	0.158	20.754
	ERR	9.9922e-07	1.1714e-07	7.7419e-07
$n = 800$				
$m = 750$	IT	53(16)	1(3)	5(12)
$p = 50$	CPU	1.490	0.313	47.016
	ERR	9.9979e-07	8.8309e-08	9.4872e-07
$n = 1000$				
$m = 950$	IT	56(3)	1(2)	5(24)
$p = 50$	CPU	1.750	0.538	91.696
	ERR	9.99794e-07	7.08476e-08	8.47506e-07

HSS preconditioning, respectively. The parameter α in the $\mathcal{P}_{LSS}$ is taken the same as for the $\mathcal{P}_{HSS}$. IT, CPU and ERR stand for the iteration numbers, the elapsed CPU times (in seconds) and the relative error, respectively. To demonstrate more efficient of local shift-splitting method, the HSS preconditioner is considered here. As one can see in Table 3, for which numerical results are obtained with HSS preconditioner, worse than GMRES without preconditioning. For the

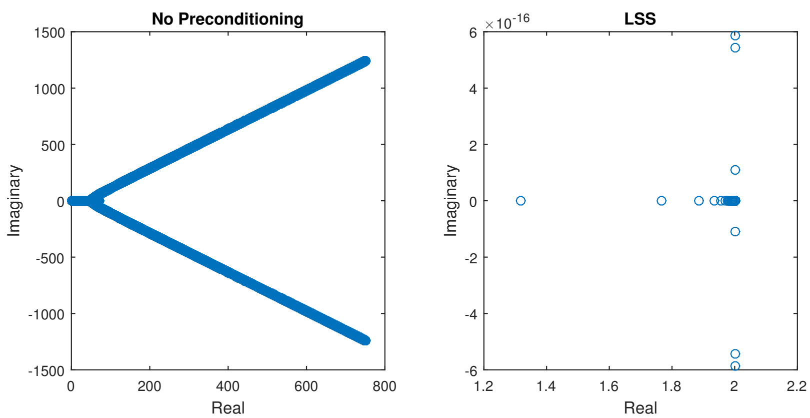

Figure 3: Eigenvalue distribution of LSS-preconditioner for Example 4.2 with $n = 1500, m = 1450, p = 50$ for $\alpha = 0.01$

second example, we depict the eigenvalue distribution of the saddle point matrix and the LSS preconditioned saddle point matrices. For $\alpha = 0.01$and $n = 1500, m = 1450, p = 50$ Figure 3 shows the eigenvalue distribution of the double saddle point matrix $\mathcal{A}$. In Figure 3 'No preconditioning' and 'LSS' stand for the original double saddle point matrix $\mathcal{A}$ and the local shift-splitting preconditioned matrix $\mathcal{P}_{LSS}^{-1}\mathcal{A}$, respectively. In fact, Figure 3 say that the eigenvalues of the LSS preconditioned matrix is more cluster than those of the double saddle point matrix $\mathcal{A}$.

5 Conclusion

In this paper we introduced and analyzed local shift-splitting iterative method and local shift-splitting precondition for solving large and sparse linear systems in double saddle point form. Convergence analysis of LSS iterative method implies that the given LSS method is convergent unconditionally. The proposed local shift-splitting precondition is able to achieve fast convergence when applied to Krylov subspace method, like GMRES. The eigenvalue distribution of the preconditioned saddle point matrix and its spectral properties are also given. Some numerical experiments show that the eigenvalues of the LSS preconditioned matrix is more cluster than those of the double saddle point matrix $\mathcal{A}$ and they have fast convergence rate.

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