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Some Results on Gelfand Paires

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Abstract

A Gelfand pair is a pair (G, K) consisting of a group G and a subgroup K (called an Euler subgroup of G) that satisfies a certain property on restricted representations. When G is a locally compact topological group and K is a compact subgroup, (G, K) is a Gelfand pair if and only if the algebra of $(K, K)-$double invariant compactly supported continuous functions(measures) on G with multiplication defined by convolution is commutative. In studying the concept of Gelfand pairs, the identification of spherical functions is of particular importance. In this paper, the spherical functions of Gelfand pair (G, K) in subspace E_1 of $\mathrm{L}^1(\mathrm{G})$ containing functions of form $f * \tilde{f}$ is introduced, where f belonges to $C_c(G)$(The convolution algebra of continuous, complex-valued functions on G with compact support). Also the characters of $E_1^{\#}$ have been identified. Finally, by introducing the space $\widehat{G}_{\#}$ including the bi-K-invariant unitary characters and the space $\widehat{G_{\#}}$ including bounded spherical functions, the locally compact groups G relatively to $\widehat{G}_{\#}=\widehat{G_{\#}}$, are characterized.

1 Introduction

Let G be a locally compact group with left Haar measure dx and let $C_c(G)$ be the convolution algebra of continuous, complex-valued functions on G with compact support, and suppose that K be a compact subgroup of G with normalized Haar measure dk. $C_c^{\#}(G)$ is the space of functions in $C_c(G)$ which are bi-invariant with respect to K, i.e functions f which satisfy

$$f(kxk') = f(x) \qquad (x \in G; k, k' \in K).$$

The space $C_c^{\#}(G)$ is a subalgebra of the convolution algebra $C_c(G)$. The pair (G,K) is said to be Gelfand pair if the convolution algebra $C_c^{\#}(G)$ is commutative. The theory of Gelfand pairs is closely related to the topic of spherical functions in the classical theory of special functions, and to the theory of Riemannian symmetric spaces in differential geometry. Our goal in this article is to characterize the locally compact groups G such that the set of bounded spherical functions and the set of bi-K-invariant unitary characters of G coincide. In the following section, we recall the basic definitions, propositions and theorems related to Gelfand pairs. In section 3, we look for $h \in C_c(G)$ such that $h * \tilde{h}$ is spherical function, then we find characters of $E_1^{\#}$ containing functions of E_1 which are bi-invariant with respect to K . In section 4, we state the main theorems of this article to characterize the locally compact groups G such that every bounded spherical function is a bi-K-invariant unitary character of G. We also bring some of examples. In the final section, we examine an important property of locally compact groups G, in which (G, K) is a Gelfand pair.

2 Gelfand pair and spherical functions

In this section, we recall the basic definitions, propositions and theorems related to Gelfand pairs. Suppose that G is a locally compact group with left Haar measure dx and let $C_c(G)$ be the convolution algebra of continuous, complex-valued functions on G with compact support, where multiplication defined by convolution is as follows

$$f \star g(x) = \int_G f(y) g(y^{-1}x) dy.$$

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Also let K be a compact subgroup of G with normalized Haar measure dk. For every $f \in C_c(G)$, the projection of f is defined by

$$f^{\#}(x) = \int_K \int_K f(kxk')dkdk'$$

It is easy to see that

$$f^{\#}(kxk') = f^{\#}(x) \qquad (x \in G; k, k' \in K),$$

hence $f^{\#}$ is in $C_c^{\#}(G)$.

Definition 2.1. *The pair (G, K) is said to be a Gelfand pair if the convolution algebra $C_c^{\#}(G)$ is commutative.*

Proposition 2.2. [1, proposition 6.1.2] *Let (G, K) be a Gelfand pair. Then G is unimodular.*

The content of the following proposition provides a structural method for identifying Gelfand pairs.

Proposition 2.3. [1, proposition 6.1.3] *Let G be a locally compact group and K a compact subgroup of G. Assume there exists a continuous involutive automorphism θ(i.e. $\theta^2 = I$) of G such that*

$$\theta(x) \in Kx^{-1}K$$

for all $x \in G$. Then (G, K) is a Gelfand pair.

Let (G, K) be a Gelfand pair.

Definition 2.4. *Let φ be a continuous, bi-K-invariant function on G. The function φ is called a spherical function if the functional χ defined by*

$$\chi(f) = \int_G f(x)\varphi(x^{-1})dx$$

is a non-trivial character of the convolution algebra $C_c^{\#}(G)$, i.e.

$$\chi(f \star g) = \chi(f).\chi(g) \tag{2.1}$$

for all f,g belong to $C_c^{\#}(G)$.

Theorem 2.5. *[1] Let φ be a continuous function on G, bi-K-invariant, $\varphi \neq 0$. Then φ is a spherical function if and only if for all x, $y \in G$*

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y). \tag{2.2}$$

In particular, $\varphi(e) = 1$.

Theorem 2.6. *[1] Let φ be a bi-K-invariant, continuous function on G. The function φ is spherical if and only if*

(I) *$\varphi(e)$=1,*

(II) *for every f belong to $C_c^{\#}(G)$ there is a complex number $\chi(f)$ such that $f \star \varphi$=$\chi(f)\varphi$.*

Theorem 2.7. *[1] Let φ be a bounded spherical function. The mapping*

$$f \mapsto \chi(f) = \int_G f(x)\varphi(x^{-1})dx$$

is a character of $L^1(G)^{\#}$, and each non-trivial character of $L^1(G)^{\#}$ is of this form.

Definition 2.8. *A representation of G is a pair $(\pi, \mathscr{H})$ of a Hilbert space $\mathscr{H}$ and a homomorphism $\pi : G \to GL(\mathscr{H})$ such that the mapping $(x, v) \to \pi(x)v$ from $G \times \mathscr{H}$ to $\mathscr{H}$ is continuous.*

We calls $(\pi, \mathscr{H})$ is an unitary representation if $\pi(x)$ is an unitary operator for all $x \in G$, i.e. $\pi(x)\pi(x)^* = \pi(x)^*\pi(x) = I$. A linear subspace $\mathscr{H}_1$ of $\mathscr{H}$ is called invariant if $\pi(x)\mathscr{H}_1 \subseteq \mathscr{H}_1$ for all $x \in G$. A representation $(\pi, \mathscr{H})$ of G is called irreducible if the only closed invariant subspaces of $\mathscr{H}$ are 0 and $\mathscr{H}$ itself . Finally, every vector $v \neq 0$ in $\mathscr{H}$ is said to be cyclic, if the closure of span$(\pi(G)v)$ is $\mathscr{H}$, i.e. $\mathscr{H} = \overline{span_{x \in G}\pi(x)v}$.

Definition 2.9. *A complex-valued locally integrable function φ is said to be positive-definite if*

$$\int_G \int_G \varphi(x^{-1}y) f(x) \overline{f(y)} dx dy \geq 0$$

for all $f \in C_c(G)$.

According to Theorem 5.1.6 of [1], any bounded continuous positive-definite function φ is of form

$$\varphi(x) = \langle \varepsilon, \pi(x)\varepsilon \rangle, \qquad x \in G,$$

where π is an unitary representation of G on a Hilbert space $\mathscr{H}$ and $\varepsilon \in \mathscr{H}$. A continuous positive-definite function φ on G with $\varphi(e) = 1$ is said to be elementary, if the associate unitary representation of G is irreducible. Let $\mathscr{H}_e$ denote the space of vectors in $\mathscr{H}$ fixed under the K-action, i. e.

$$\mathscr{H}_e = \{v \in \mathscr{H} : \pi(k)v = v, \forall k \in K\}.$$

By lemma 6.2.2 of [1], the cyclic vector ε belongs to $\mathscr{H}_e$ if and only if φ is bi-K-invariant. Finally, an irreducible unitary representation π of G on a Hilbert space $\mathscr{H}$ is said to be of class one if the subspace of K-fixed vectors $\mathscr{H}_e$ is non-trivial. By corollary 6.3.3 of [1], the positive-definite spherical functions on G correspond one-to-one to the equivalence classes of irreducible unitary representations of G of class one.

Theorem 2.10. *[1] Let G be compact Gelfand pair.Then every spherical function is positive-definite,and so any irreducible unitary representation of compact groups is finite-dimensional.*

Theorem 2.11. *[1] Let φ be a continuous, positive-definite function, that is bi-K-invariant, and satisfies $\varphi(e) = 1$. Then φ is a (positive-definite) spherical function if and only if φ is elementary.*

3 The space of characters of $E_1^{\#}$

In this section (G, K) is a Gelfand pair. Assume that

$$\begin{aligned} E_1 &= span\{f \star \tilde{g} : f, g \in C_c(G)\}, \\ E_2 &= span\{f \star \tilde{f} : f \in C_c(G)\}, \end{aligned}$$

where $\tilde{f}(x) = \overline{f(x^{-1})}$. It is clearly that E_1 and E_2 are the linear subspaces of $L^1(G)$. Also it is easy to see that $E_1 = E_2$. In fact, in terms of equality

$$f \star \tilde{g} = 1/4((f+g) \star \widetilde{(f+g)} - (f-g) \star \widetilde{(f-g)} + i(f+ig) \star \widetilde{(f+ig)} - i(f-ig) \star \widetilde{(f-ig)}),$$

this is clear.

Our goal in this section is to investigate and identify spherical functions in E_1. We will also identify characters of $E_1^{\#}$.

3.1 Spherical functions in E_1

To investigate the spherical functions in E_1, suppose that $h \star \tilde{h} \in E_1$, where $h \in C_c(G)$. In this case we will have

$$\begin{aligned} h \star \tilde{h}(x) &= \int_G h(y) \tilde{h}(y^{-1}x) dy = \int_G h(y) \overline{h}(x^{-1}y) dy \\ &= \int_G h(y) \overline{L_x h}(y) dy = \langle h, L_x h \rangle. \end{aligned} \tag{3.1}$$

where L_x is the left traslation operator defined by $L_x h(y) = h(x^{-1}y)$, and

$$\langle h, g \rangle = \int_G h(y) \overline{g}(y) dy$$

is inner product of $L^2(G)$. But we know that for every $h \in C_c(G)$, $h \star \tilde{h}$ is a bounded, continuous positive-definite function. Hence there is a unitary representation π of G on some Hilbert space $\mathscr{H}$, such that

$$h \star \tilde{h}(x) = \langle \varepsilon, \pi(x)\varepsilon \rangle, \qquad (x \in G)$$

where ε is cyclic vector for π in $\mathscr{H}$.

On the other hand, the relation 3.1 shows that the representation π probably is L_x. But it is possible that h is not cyclic. In other words, it is possible that the vectors $L_x h$, where x runs through G, does not span a dense subspace of $L^2(G)$. Set $\mathscr{H} = \overline{\underset{x\in G}{span} L_x h}$. Clearly $\mathscr{H}$ is a closed subspace of $L^2(G)$, and hence is a Hilbert space. Moreover $\mathscr{H}$ is invariant under L. Indeed

$$L_s L_x h = L_{sx} h, \qquad (s, x \in G)$$

Therefore we define:

$$\pi : G \to End\mathscr{H}, \qquad \pi(x) = L_x : \mathscr{H} \to \mathscr{H}.$$

Clearly π is an unitary representation of G on $\mathscr{H}$ and so $h = L_e h$ is cyclic relatively to π because $\overline{\underset{x\in G}{span}\pi(x)h} = \overline{\underset{x\in G}{span} L_x h} = \mathscr{H}$. Hence

$$h \star \tilde{h}(x) = \langle h, L_x h \rangle = \langle h, \pi(x)h \rangle.$$

Thus the unitary representation of G associated with continuous positive-definite function $h * \tilde{h}(x)$ is π, where h is cyclic. Note that π is irreducible. Indeed for each element $L_{x_0} h \in \mathscr{H}$, we have

$$\overline{\underset{x\in G}{span} L_x L_{x_0} h} = \overline{\underset{x\in G}{span} L_{xx_0} h} = \mathscr{H}$$

Therefore for each compact group G, $\mathscr{H}$ can not be equal to $L^2(G)$, because in this case the irreducible unitary representation $\pi = L$ will be finite-dimensional, i.e. the space $L^2(G)$ is finite-dimensional that is not true.

Now we look for $h \in C_c(G)$ such that $h \star \tilde{h}$ is a spherical function:

Theorem 3.1. *Suppose that $h \in C_c(G)$ satisfies in following conditions:*
(I) $\| h \|_2 = 1$,
(II) for every $k \in K$, $x \in G$: $\quad h(kx) = h(x)$.

Then $h \star \tilde{h}$ is a (positive-definite)spherical function.

Proof. Proof. The condition (I) shows that:

$$h \star \tilde{h}(e) = \langle h, \pi(e)h \rangle = \langle h, h \rangle = \| h \|_2^2 = 1$$

Also by the condition (II), $h \star \tilde{h}$ is bi-K-invariant, since $h \in \mathscr{H}_e$. (The space of vectors in $\mathscr{H}$ fixed under K-action.) Since unitary representation relative to continuous, positive-definite function $h \star \tilde{h}$ is irreducible, Hence $h \star \tilde{h}$ is elementary. Thus by the theorem 2.11 is a spherical function. □

Remark 3.2. *For each h with above conditions, because $h \star \tilde{h}$ is bounded, therefore is bounded spherical function, hence by Theorem 2.7, the mapping*

$$f \mapsto \chi(f) = \int_G f(x) h \star \tilde{h}(x^{-1}) dx$$

is a character of $L^1(G)^{\#}$. Also since $h \in \mathscr{H}_e$, $\mathscr{H}_e \neq 0$. Hence π is of class one, so by proposition 6.4.1 *of [1], $h \star \tilde{h}$ is the extremal point of $\mathcal{P}_0^{\#}$, where $\mathcal{P}_0^{\#}$ is set of continuous, bi-K-invariant, positive-definite functions φ on G with $\varphi(e) \leq 1$.*

Remark 3.3. *For every h as above, because $h \star \tilde{h}$ is spherical , by theorem 2.6,*

$$\forall f \in C_c^{\#}(G), \quad \exists \chi(f) \in \mathbb{C} \quad s.t \qquad f \star (h \star \tilde{h}) = \chi(f) h \star \tilde{h}.$$

Now we set $S = span\{h\star\tilde{h} : h \in C_c(G), h\star\tilde{h}$ is spherical$\}$. S is subalgebra of convolution algebra $C_c^{\#}(G)$. Indeed the convolution product $\star$ in S is closed, because for both members of S, $h\star\tilde{h}$ and $g\star\tilde{g}$ we have

$$\exists\chi(h\star\tilde{h}) \quad s.t \qquad (h\star\tilde{h})\star(g\star\tilde{g}) = \chi(h\star\tilde{h})g\star\tilde{g}\in S.$$

In general case for each member $s=\sum_{i=1}^{n}\alpha_i h_i\star\tilde{h_i}$ of S we have

$$\forall f\in C_c^{\#}(G), \qquad f\star s = \sum_{i=1}^{n}\alpha_i f\star(h_i\star\tilde{h_i}) = \sum_{i=1}^{n}\alpha_i\chi_{h_i}(f)h_i\star\tilde{h_i}\in S$$

In fact S is the left ideal in $C_c^{\#}(G)$.

3.2 Characters of $E_1^{\#}$

Suppose that (G,K) be a Gelfand pair. Therefore by proposition 2.2, G is unimodular and so $\triangle$=1, where $\triangle$ is the Haar modulus of the group G. Hence

$$f^*(x) = \triangle(x)^{-1}\overline{f(x^{-1})} = \overline{f(x^{-1})} = \tilde{f}(x).$$

According to $(f\star g)^* = g^*\star f^*$, we have $\widetilde{f\star g} = \tilde{g}\star\tilde{f}$, $\widetilde{g\star\tilde{g}} = g\star\tilde{g}$ and also for members $f\star\tilde{f}$, $g\star\tilde{g}\in E_1$ we have

$$(f\star\tilde{f})\star(g\star\tilde{g}) = (f\star\tilde{f})*\widetilde{g\star\tilde{g}},$$

that is in $E_1 = E_2$. Thus E_1 is a subalgebra of $C_c(G)$. For the same reason $E_1^{\#}$ is the subalgebra of $C_c^{\#}(G)$. But each $f\in C_c(G)$ could be approximated by approximate unit. Indeed we can approximate f by $f\star u_V$, where V is a neighbourhood of e and $u_V\in C_c(G)$ is approximate unit. Therefore according to $f\star u_V = f\star\widetilde{\widetilde{u_V}}$, each $f\in C_c(G)$ is approximated by the functions such as $f\star\tilde{g}$, hence E_1 is in $C_c(G)$ and so is dense in $L^1(G)$. According to the following lemma we can also prove that $E_1^{\#}$ is dense in $L^1(G)^{\#}$.

Lemma 3.4. *Suppose that $h\in C_c(G)$. Set*

$$h_K(x) = \int_K h(kx)dk, \qquad (x\in G)$$

Then h_K is left-K-invariant and so is a continuous function with compact support such that

$$(h\star\tilde{h})^{\#} = h_K\star\widetilde{h_K}. \tag{3.2}$$

Proof. Proof. By lemma 4.3.1 of [1], h_K is continuous function. Since h_K vanishes outside $\pi_K(suppf)$, it has compact support, thus $h_K\in C_c(G)$. On the other hand, by change of variable $k\mapsto kk'^{-1}$ we observe

$$h_K(k'x) = \int_K h(kk'x)dk = \int_K h(kx)dk = h_K(x).$$

Hence h_K is left-K-invariant. Finally, we get

$$h_K\star\widetilde{h_K}(x) = \int_G h_K(y)\overline{h_K(x^{-1}y)}dy = \int_G\int_K\int_K h(ky)\overline{h(k'x^{-1}y)}dkdk'dy$$

Now by $Fubini's$ theorem and change of variable $y\mapsto k^{-1}y$ we get

$$h_K\star\widetilde{h_K}(x) = \int_K\int_K\int_G h(y)\overline{h(k'x^{-1}k^{-1}y)}dydkdk' = \int_K\int_K\langle h, L_{kxk'^{-1}}h\rangle dkdk'$$

So finally with change of variable $k'\mapsto k'^{-1}$ we have

$$h_K\star\widetilde{h_K}(x) = \int_K\int_K h\star\tilde{h}(kxk'^{-1})dkdk' = (h\star\tilde{h})^{\#}(x).$$

□

Remark 3.5. *According to the above lemma the operation # is closed in E_1. Also, $h_K \star \widetilde{h_K}(x)$ is bi-K-invariant, since h_K satisfies in conditions of theorem 3.1. Of course if necessary, we replace h_K by $\frac{h_K}{\|h_K\|_2}$.*

Lemma 3.6. *$E_1^{\#}$ is dense in $L^1(G)^{\#}$.*

Proof. Proof. Let $f \in L^1(G)^{\#}$. Then $f \in L^1(G)$. Because of density of E_1 in $L^1(G)$ there exists a sequence $f_n \in E_1$ such that $\lim_{n\to\infty} \| f_n - f \|_1 = 0$. But $f_n = \sum_{i=1}^{\alpha_n} t_i h_i \star \tilde{h_i}$ and so

$$f_n^{\#} = \sum_{i=1}^{\alpha_n} t_i (h_i \star \tilde{h_i})^{\#} = \sum_{i=1}^{\alpha_n} t_i h_{iK} \star \widetilde{h_{iK}} \in E_1^{\#}.$$

Because of $f = f^{\#}$ we have

$$\| f_n^{\#} - f \|_1 = \| f_n^{\#} - f^{\#} \|_1 = \| (f_n - f)^{\#} \|_1 \leq \| f_n - f \|_1 .$$

Hence by assumption $\lim_{n\to\infty} \| f_n^{\#} - f \|_1 = 0$. Since $f_n^{\#} \in E_1^{\#}$, the lemma follows. □

Corollary 3.7. *According to the above lemma, each non-trivial character from $E_1^{\#}$ is extended to a non-trivial character of $L^1(G)^{\#}$. Therefore according to Theorem 2.7, it will be as follows*

$$f \mapsto \chi(f) = \int_G f(x)\varphi(x^{-1})dx,$$

where φ is a spherical function. Hence the space of characters of $E_1^{\#}$ is the set of bounded spherical functions.

4 Relation between characters of $E_1^{\#}$ and bi-K-invariant unitary charactrers of G

We assume (G, K) is a Gelfand pair.
Let $\widehat{G}_{\#}$ be set of bi-K-invariant unitary characters of G. And also let $\widehat{G_{\#}}$ be set of bounded spherical functions(the space of characters of $E_1^{\#}$).
Our goal in this section is to characterize the locally compact groups G such that $\widehat{G}_{\#} = \widehat{G_{\#}}$. At first we note that if $\chi \in \widehat{G}_{\#}$, then

$$\chi(xy) = \chi(x)\chi(y), \quad \chi(x^{-1}) = \overline{\chi(x)}, \quad \chi(e) = 1.$$

And

$$\chi(kxk') = \chi(x), \qquad (x \in G; k, k' \in K).$$

Now for $x = k' = e$, we have $\chi(k) = \chi(e) = 1$, where k runs through K. Therefore

$$\int_K \chi(xky)dk = \int_K \chi(x)\chi(k)\chi(y)dk = \int_K \chi(x)\chi(y)dk = \chi(x)\chi(y),$$

Since $\int_K dk = 1$. Hence by theorem 2.5, χ is spherical function, so $\chi \in \widehat{G_{\#}}$, i.e. $\widehat{G}_{\#} \subseteq \widehat{G_{\#}}$. But this inclusion may be pure. Indeed suppose that φ is spherical function of the compact Gelfand pair (G, K). Since G is compact, by theorem 6.5.1 0f [1], φ is positive-definite and so unitary representation associated with spherical function φ (i.e. $(\pi, \mathscr{H})$) is finite-dimensional. And also

$$\int_G |\varphi(x)|^2 dx = \frac{1}{d},$$

where d is the dimension of $\mathscr{H}$.

Now if $d > 1$ then φ can not be character of G else $|\varphi| = 1$ and so $d = 1$. (note that dx is normalized Haar measure of G, i.e. $\int_G dx = 1$).

Example 4.1. [1, Chapter 7] *Let $G = SO(3)$, $K = SO(2)$. It has shown that (G, K) is compact Gelfand pair, and spherical functions of pair (G, K) are zonal functions $\varphi_n = \frac{Z_n}{Z_n(e_1)}$, and also unitary representation associated with spherical functions have dim$= 2n + 1 > 1$, hence $\widehat{G}_{\#} \subsetneq \widehat{G_{\#}}$.*

Example 4.2. *Let $G = \mathbb{R}$ be the additive group of real numbers under the Euclidean topology and let $K = \{0\}$ which is obviously a compact subgroup of it . Obviously (G, K) is a Gelfand pair, and spherical functions of this pair are*

$$\varphi(x) = e^{\lambda x} \qquad (\lambda \in \mathbb{C}),$$

where λ is a constant complex number. Therefore the bounded spherical functions are $\varphi(x) = e^{i\theta x}$, where $\lambda = i\theta$. But these are the same unitary characters of G, hence $\widehat{G}_{\#} = \widehat{G_{\#}}$.
In example 4.2, G is abelian. In the general case it is easily seen that

Theorem 4.3. *Let G be abelian locally compact group, and K be a compact subgroup of G with normalized Haar measure dk. Then (G, K) is Gelfand pair and $\widehat{G}_{\#} = \widehat{G_{\#}}$.*

Proof. Proof. Since G is abelian Group, $L^1(G)$ is commutative, and so is $C_c^{\#}(G)$.
Now suppose that $\varphi \in \widehat{G_{\#}}$, therefore φ is bounded spherical, so by assumption is bi-K-invariant and also by theorem 2.5 we have

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y). \qquad (x, y \in G)$$

Since G is abelian Group, the left side of the equality is equal to

$$\int_K \varphi(xky)dk = \int_K \varphi(kxy)dk = \int_K \varphi(xy)dk = \varphi(xy).$$

Therefore we obtain

$$\varphi(xy) = \varphi(x)\varphi(y), \qquad (x, y \in G).$$

Hence φ is a continuous homomorphism of G. But since φ is bounded function, it is unitary. Indeed if there is $x \in G$ such that $|\varphi(x)| < 1$, then

$$|\varphi(x^{-1})| = \frac{1}{|\varphi(x)|} > 1.$$

Hence $|\varphi(x^{-n})| = |\varphi(x^{-1})^n| = |\varphi(x^{-1})|^n \to \infty$, when $n \to \infty$. We get a contradiction. Similarly the case of $|\varphi(x)| > 1$ is false. Therefore $|\varphi(x)| = 1$ for all $x \in G$ and so $\varphi \in \widehat{G}_{\#}$, i.e. $\widehat{G}_{\#} = \widehat{G_{\#}}$. □

Now we proceed with the case that G is nonabelian.

Theorem 4.4. *Let (G, K) be compact Gelfand pair such that G is nonabelian Group. Suppose that all irreducible unitary representations of class one are one-dimensional. Then all spherical functions are unitary characters of G, i.e. $\widehat{G}_{\#} = \widehat{G_{\#}}$.*

Proof. Proof. Let φ be a spherical function of the pair (G, K). Since G is compact, φ is positive-definite and so there is irreducible unitary representation $(\pi, \mathscr{H})$ of G such that

$$\varphi(x) = \langle \varepsilon, \pi(x)\varepsilon \rangle, \qquad (x \in G),$$

where ε a cyclic vector in $\mathscr{H}$. But by assumption π is one-dimensional, hence $\pi(x) = \chi(x)I$, where χ is a unitary character of G. Moreover, since $\| \varepsilon \| = 1$, we have

$$\varphi(x) = \langle \varepsilon, \pi(x)\varepsilon \rangle = \langle \varepsilon, \chi(x)I\varepsilon \rangle = \langle \varepsilon, \chi(x)\varepsilon \rangle = \overline{\chi(x)}\langle \varepsilon, \varepsilon \rangle = \overline{\chi(x)}, \quad (x \in G).$$

Therefore φ is unitary character of G, i.e. $\widehat{G}_{\#} = \widehat{G_{\#}}$. □

Corollary 4.5. *Let (G, K) be compact Gelfand pair such that G is nonabelian Group. If there is an irreducible unitary representation of class one with dimensional larger than 1, then the corresponding spherical function is not unitary character of G, i.e. $\widehat{G}_{\#} \subsetneqq \widehat{G_{\#}}$.*

Example 4.6. *Suppose that G_1 be compact abelian group and so $K = G_2$ be compact nonabelian group. Then $G = G_1 \times G_2$ is compact nonabelian group such that (G, K) is Gelfand pair. Indeed it is easily seen that the mapping θ by $\theta(x) = x^{-1}$ is a continuous involutive automorphism of G such that $\theta(x) \in Kx^{-1}K$ for all $x \in G$. And also all spherical representations(i.e. the irreducible unitary representations of class one) are one-dimensional, hence by theorem 4.4, $\widehat{G}_{\#} = \widehat{G_{\#}}$.*

Example 4.7. *Let $G = SU(2)$, $K = S(U(1) \times U(1))$. Then (G, K) is Gelfand pair. It has been proven dimension of irreducible unitary representations of G are $1, 2, 3, \ldots$. It can be shown that the only representations with odd dimension are of class one, hence there is one irreducible unitary representation of class one with $\dim > 1$. Therefore by corollary 4.5, $\widehat{G}_\# \subsetneqq \widehat{G_\#}$.*

Remark 4.8. *According to the example 4.6, for each the compact nonabelian group G, if we can write $G = G_1 \times G_2$, where G_1 be compact abelian group and $K = G_2$ denote the compact nonabelian group. Then (G, K) is Gelfand pair and also $\widehat{G}_\# = \widehat{G_\#}$.*

Theorem 4.9. *(Sophus Lie). Every irreducible finite-dimensional representation of a solvable Lie group G has dimension one.*

Theorem 4.10. *Let (G, K) be Gelfand pair, where G is solvable compact Lie group. Then $\widehat{G}_\# = \widehat{G_\#}$.*

Proof. Proof. Since G is compact, every irreducible unitary representation of G is finite-dimensional, hence by theorem 4.9, is one-dimensional. Therefore G is abelian group and so by theorem 4.3, $\widehat{G}_\# = \widehat{G_\#}$. □

Theorem 4.11. *Let (G, K) be Gelfand pair. If $\widehat{G}_\# = \widehat{G_\#}$ then all irreducible unitary representations of G of class one are one-dimensional.*

Proof. Proof. By corollary 6.3.3 of [1], the positive-definite spherical functions on G correspond one-to-one to the equivalence classes of irreducible unitary representations of G of class one. Hence by assumption the positive-definite spherical functions(that are bounded) are bi-K-invariant unitary characters of G, and so all irreducible unitary representations of class one are one-dimensional. □

Finally, from theorems 4.4 and 4.11 we have:

Corollary 4.12. *Let (G, K) be compact Gelfand pairs. Then $\widehat{G}_\# = \widehat{G_\#}$ if and only if all irreducible unitary representations of G of class one are one-dimensional.*

5 The other property of Gelfand pairs

In this section, we consider an important property of locally compact groups that is important in the theory of group representations. This property is based on the approximation of the constant function 1 on compact sets by elements of E_1:

Definition 5.1. [2, Definition 8.3.14] *A locally compact group G has the property P' if the constant function 1 can be approximated uniformly on compact sets by functions of the form $h \star \tilde{h}$, where $h \in C_c(G)$.*

The basic and important point is that this property is equivalent to the property P_1 defined in [2]. If G has the property P' then by proposition 8.3.17 of [2], every continuous positive-definite function on G can be approximated uniformly on compact sets by functions of the form $h \star \tilde{h}$, with $h \in C_c(G)$. Our goal in this section is to investigate the conditions on Gelfand pairs under which G has this property.

Let P be the set of all continuous, positive -definite functions on G. Let Q be the closure in the compact-open topology on P of $C_c(G) \cap P$. According to lemma 4.1 of [4], we know Q is the closure in the compact-open topology on P of the set of all functions of the form $h * \tilde{h}$ with h in $C_c(G)$.
Now it is known [3] that 1 is in Q if and only if $Q = P$. But $1 \in Q$ is the same property P' of G ([2]), i.e. the constant function 1 can be approximated uniformly on compact sets by functions of the form $h * \tilde{h}$, with $h \in C_c(G)$. For G unimodular is equivalent to $\int_G f(x)dx \geqq 0$ for all f in $P \cap L^1(G)$.

Let (G, K) be Gelfand pair, hence G is unimodular. Our goal in this section is to characterize some of the Gelfand pairs (G, K) such that G has the property P'. In other words $Q = P$.

Theorem 5.2. *If H is a closed normal subgroup of a locally compact group G and if P' holds for H and for G/H, then P' holds for G.*

Proof. cf. Theorem 8.3.3 of [2]. □

Theorem 5.3. *Let (G, K) be a Gelfand pair. If K is normal subgroup of G, then G has the property P'.*

Proof. Proof. Since K is compact, P' holds for K, and also since (G, K) is Gelfand pair, $L^1(G)^{\#}$ is commutative. But by assumption G/K is a locally compact topological group, so since $L^1(G)^{\#} = L^1(G/K)$ is commutative , G/K is abelian group, therefore P' holds for G/K (see [2]). Hence according to theorem 5.2, G has the property P'. □

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