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# A new model for evaluating cost efficiency using output slack variables in Data Envelopment Analysis

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**Abstract**

In previous methods to estimate cost efficiency using data envelopment analysis, a dominated decision-making unit (DMU) in the production possibility set(PPS) can be introduced as a cost efficiency unit. That is, a DMU may be cost-efficient, but not performance efficient. In this paper, at first, we provide a new definition for a cost-efficient unit, and then a model is presented in which the necessary condition for introducing a cost efficiency of DMU is its parato efficiency.

## 1 Introduction

Data envelopment analysis is a non-parametric method for evaluating the efficiency of similar Decision-Making Units (DMU). It is to use the same inputs to produce the same outputs [3, 1]. Cost efficiency (CE) is one of the concepts of data envelopment analysis that measures the ability of a DMU to produce current output at the lowest cost. In such cases, the price of each input is available. Cost efficiency was first proposed by Farrell [3] and then developed by Farr et al. [5], using linear programming. The proposed model requires the input and output values of each DMU and also the price of each input. Tone [7] expressed the shortcomings of the cost efficiency model when the price of each input is different for each DMU, and proposed a new method. The proposed method creates a set called Cost Space(CS) by multiplying the inputs of each DMU in its corresponding price and then evaluates the DMUs in this space. In all previous methods for evaluating cost-efficiency, a dominated DMU in the production possibility set (PPS) can also be introduced as a cost-efficient unit. In other words, may a DMU be cost-efficient, but not efficient in performance. It seems logical that a DMU should be called cost-efficient, which is also performance efficient. It means, not only the cost-efficient DMU works well, but it also produces its outputs at the lowest cost.

In this paper, we provide an example to illustrate this point and then propose a new definition of cost-efficiency. Using this definition and the models that follow, we show that Parato efficiency is a necessary condition for the cost efficiency of each DMU.

At first, we state the Farr et al [5] method to evaluate cost efficiency and its shortcomings. In the second section, we define a new definition and method for evaluating cost efficiency. Then, using the stated approach, a model for evaluating cost efficiency is presented. The fourth section will be included with a numerical example and finally, the results will be shown.

## 2 Farr et. al [5] method for evaluating CE

Suppose there are $n$ DMUs under evaluation with the input and output $(X_j, Y_j)$, $(j = 1, 2, \cdots, n)$ respectively, that $X_j = (x_{1j}, \cdots, x_{mj}) \in \Re^m$ and $Y_j = (y_{1j}, \cdots, y_{sj}) \in \Re^s$.

The Production Possibility Set with variable returns to scale $(PPS_V)$, is expressed as follows:

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$$PPS_V = \{(X,Y) : X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \ (j = 1, \cdots, n)\}$$

The BCC input-oriented (BCC-I) model evaluates the efficiency of $DMU_p$, by solving the following linear program:

Min $\theta_p$
s.t. $\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_p x_{ip}, i = 1 \cdots, m,$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad r = 1, \cdots, s, \qquad (\mathbf{1})$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad j = 1, \cdots, n.$$

where $x_{ij}$ and $y_{rj}$ (all nonnegative) are the inputs and outputs of the $j$th DMU.

**Definition 2.1.** *$DMU_p$ is called technical efficient, if in model ($\mathbf{1}$) we have $\theta_p^* = 1$, where $(*)$ is the optimum sign.*

**Definition 2.2.** *$DMU_p$ is called BCC-strong efficient, if it is technical efficient and in any optimal solution of model ($\mathbf{1}$), all slack variables are equal to zero.*

**Definition 2.3.** *$DMU_p$ is Parato efficinet in $PPS_V$, if it is not dominated by any other DMU. That means there is no other DMU in $PPS_V$ with the property that all inputs are less than or equal to the inputs of $DMU_p$ and all outputs are greater than or equal to the outputs of $DMU_p$ and the inequality is strictly in at least one component.*

**Theorem 2.4.** *$DMU_p$ is Parato-efficient in $PPS_V$ if and only if it is BCCefficient.*

To evaluate the cost-efficiency of the $DMU_p(CE_p)$, Farr et al. [5] proposed the following linear programming model. By solving this model for each DMU, the lowest cost consumed to produce outputs at least as much as current outputs is obtained, by knowing the price of inputs.

$$Min \sum_{i=1}^{m} C_i x_i$$

s.t. $\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, \ i = 1, \cdots, m,$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \ r = 1, \cdots, s, \qquad (\mathbf{2})$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, j = 1, \cdots, n, \ x_i \geq 0, \ i = 1, \cdots, m.$$

where $C_i$ is input price of $i$ $(x_i)$. The optimal solution of model ($\mathbf{2}$), $x_i^*$ shows the amount of input $i$ required to produce outputs at least as much as current outputs at the lowest cost. Finally, the cost-efficiency of $DMU_p$ is obtained by using cost ratio from the following relation:

$$CE_p = \frac{\sum_{i=1}^{m} C_i x_i^*}{\sum_{i=1}^{m} C_i x_{ip}}$$

Where (*) indicates the optimization in model ($\mathbf{2}$). This measure of cost efficiency is called Farrell cost efficiency.

**Definition 2.5.** *$DMU_p$ is cost-efficient, if $CE_p = 1$.*

In other words $DMU_p$ is cost-efficient, if $\sum_{i=1}^{m} C_i x_i^* = \sum_{i=1}^{m} C_i x_{ip}$. Now we will discuss the shortcoming of the mentioned method and explain the matter with an example. We show that a DMU may be cost-efficient, but not strong efficient.

**Example 1.** Consider five decision making units with one input and one output according to Table 1. Figure 1 also shows the PPS made by them. Their cost-efficiency, evaluated by Model (2) and then Equation (3), is also given in the fourth column of Table 1. The input price of these DMUs is $C_1 = 2$. Consider three DMUs 1, 2, and 3 that consume one unit in input. $DMU_2$ produces three times as much output as $DMU_1$, and $DMU_3$ produces $\frac{5}{3}$ times as much output as $DMU_2$. That means $DMU_3$ dominates both $DMU_1$ and $DMU_2$. As can be seen, the cost-efficiency of these three DMUs are equal to one, while $DMU_3$ is just strong-efficient. This is also significant from a management point of view, where a DMU can be introduced as a cost efficient unit, that also is efficient in performance.

Example.1 data

| DMU | Input x | Output y | Farell Cost Efficient |
|---|---|---|---|
| $DMU_1$ | 1 | 1 | 1 |
| $DMU_2$ | 1 | 3 | 1 |
| $DMU_3$ | 1 | 5 | 1 |
| $DMU_4$ | 2 | 7 | 1 |
| $DMU_5$ | 5 | 6 | 0.3 |

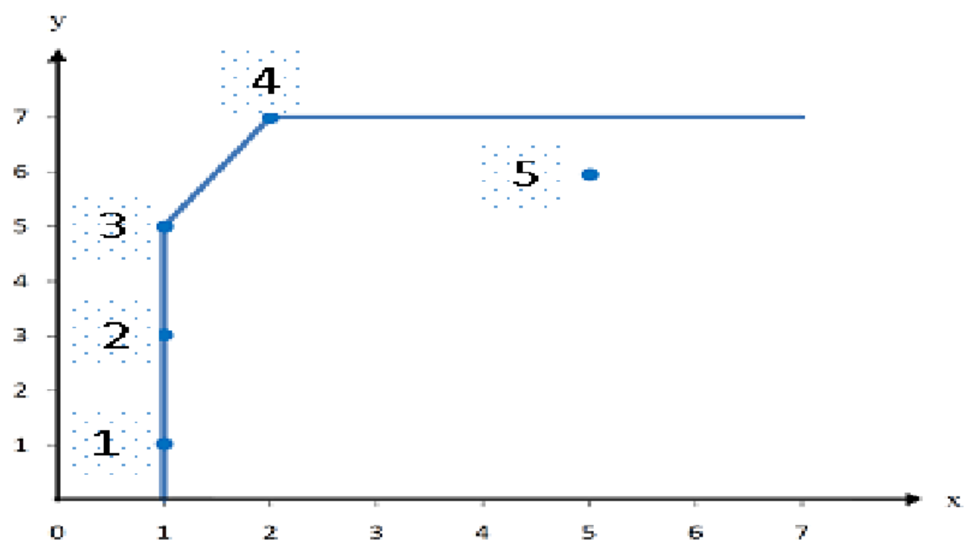


Figure 1: The Production Possibility Set of example 1

## 3 Proposed method for evaluating CE

In this section, we provide a new definition of cost efficiency and then present an approach to modify previous methods to evaluate cost-efficiency.

**Definition 3.1.** *Suppose $CE_p = 1$. We say $DMU_p$ is strong cost efficient if in each optimal solution of model* (**2**) *all slack variables are equal to zero. Otherwise $DMU_p$ is weak cost efficient.*

According to the above definition, to evaluate the cost-efficiency of a DMU, first, model (**2** ) must be solved, and then using the optimal solution of that, the following model must be solved:

$Max\overrightarrow{1}S$

$$s.t.\sum_{j=1}^{n}\lambda_j x_{ij}=x_i^*, i=1,\cdots,m,$$

$$\sum_{j=1}^{n}\lambda_j y_{rj}-s_r=y_{rp}, r=1,\cdots,s, \qquad (\mathbf{4})$$

$$\sum_{j=1}^{n}\lambda_j=1,$$

$$\lambda_j\geq 0, j=1,\cdots,n, s_r\geq 0, r=1,\cdots,s,$$

Where $x^*$ is the optimal solution of model (**2**) and $\overrightarrow{1}=(1,1,\cdots,1)$.
If in optimality of model (**4**), the value of objective function is equal to zero, then we say $DMU_p$ is strong cost-efficient. Otherwise it will be weak cost-efficient.

**Theorem 3.2.** *If $CE_p=1$, then $DMU_p$ has a technical efficiency.*

**Proof.** Suppose $CE_p=1$, then $(\lambda,X)=(e_p,X_p)$ is an optimal solution for model (**2**) in which $e_p$ is a vector that all its components are zero except the p-th component which is one. Suppose to the contrary that $\theta_p^*<1$. Since $(\theta_p^*X_p,Y_p)\in PPS$, there exists $\overline{\lambda}\geq 0$ with $\sum_{j=1}^{n}\overline{\lambda_j}=1$, $\sum_{j=1}^{n}\overline{\lambda_j}X_j\leq\theta_p^*X_p$ and $\sum_{j=1}^{n}\overline{\lambda_j}Y_j\geq Y_p$. Therefore $(\overline{\lambda}\theta_p^*X_p)$ is a possible solution of model (**2**) where $C(\theta_p^*X_p)=\theta_p^*CX_p<CX_p$. This contradicts the optimality of $(\lambda,X)=(e_p,X_p)$.

**Theorem 3.3.** *If $DMU_p$ is a strong cost-efficient, then it is strong-efficient.*

**proof.** Suppose $DMU_p$ is Strong cost-efficient, then $CE_p=1$ and in optimality of model (**4**), the optimal value of the objective function is zero. Since $CE_p=1$, $(\lambda,X)=(e_p,X_p)$ is an optimal solution for model (**2**). Now suppose to the contrary that $DMU_p$ is not strong-efficient. Thus there is a $(\overline{X},\overline{Y})\in PPS$ that dominates $DMU(p)$ i.e. $(-\overline{X},\overline{Y})\geq(-X_pY_p)$ and $(-\overline{X},\overline{Y})\neq(-X_pY_p)$. Since $(\overline{X},\overline{Y})\in PPS$, so there is $\overline{\lambda}\geq 0$ where $\sum_{j=1}^{n}\overline{\lambda_j}X_j\leq\overline{X}$, $\sum_{j=1}^{n}\overline{\lambda_j}Y_j\geq\overline{Y}$ and $\sum_{j=1}^{n}\overline{\lambda_j}=1$. If $\overline{X}\neq X_p$, then $(\overline{\lambda},\overline{X})$ will be a feasible solution of model (**2**) and $\sum_{i=1}^{m}c_i\overline{x_i})<\sum_{i=1}^{m}c_ix_{ip}$, and this contradicts the optimality of $(\lambda,X)=(e_p,X_p)$. Also, if $\overline{Y}\neq Y_{p'}$, then $(\lambda,S)=(\overline{\lambda},\overline{S})$ is a possible solution for model (**4**) where $\overline{S}=\sum_{j=1}^{n}\overline{\lambda_j}Y_j-Y_p$. Obviously in this answer $\overrightarrow{1}\overline{S}>0$ and this is also a contradiction.

Now we combine models (**2**) and (**3**) and present the following model:

Min $\sum_{i=1}^{m}c_ix_i-\varepsilon(\overrightarrow{1}S)$
s.t. $\sum_{j=1}^{n}\lambda_jx_{ij}=x_i, i=1\cdots,m,$

$$\sum_{j=1}^{n}\lambda_j y_{rj}-s_r=y_{rp},\ r=1,\cdots,s, \qquad (\mathbf{5})$$

$$\sum_{j=1}^{n}\lambda_j=1,$$

$$\lambda_j\geq 0,\ j=1,\cdots,n,\ \ x_i\geq 0,\ \ i=1,\cdots,m,$$

Where $\varepsilon$ is a non-Archimedean number.

Using the above, we note that $DMU_p$ is strong cost-efficient if and only if in each optimal solution $(x^*,y^*)$ in model (**5**), we have $\sum_{i=1}^{m}c_ix_i^*=\sum_{i=1}^{m}c_ix_{ip}$ and $\overrightarrow{1}S^*=0$.
This section describes a method for evaluating the performance of DMUs from the cost point of view. According to the mentioned method, the output slack variables are also included in the evaluation of cost-efficiency.In this section,

this is done directly by presenting a model. Note the following model:

$$SCE_p = \text{Min } \rho_p = \frac{\frac{CX}{CX_p}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r}{y_{rp}}}$$

s.t. $\sum_{j=1}^{n}\lambda_j x_{ij} = x_i,\ i = 1,\cdots,m,$

$$x_i \le x_{ip},\ i = 1,\cdots,m,$$

$$\sum_{j=1}^{n}\lambda_j y_{rj} - s_r = y_{rp},\quad r = 1,\cdots,s, \qquad (\mathbf{6})$$

$\sum_{j=1}^{n}\lambda_j = 1,$
$\lambda_j \ge 0,\ j = 1,\cdots,n,\ x_i \ge 0,\ i = 1,\cdots,m,\ s_r \ge 0,\ r = 1,\cdots,s,$
Where $X = (x_1, x_2, \cdots, x_m)$.

**Theorem 3.4.** *$DMU_p$ is strong cost-efficient, if and only if $SCE_p = 1$.*

**Proof.** Assume $DMU_p$ is a strong cost-efficiency, according to theorem ($\mathbf{3}$) it also has strong efficiency. By definition 5 we have $CE_p = 1$ and all slack variables are equal to zero in optimal solution of model ($\mathbf{2}$).By a way of contradiction suppose $SCE_p < 1$. Two cases may occur in model ($\mathbf{6}$).
In the first case; there is optima solution such as $(\widetilde{\lambda}, \widetilde{X}, \widetilde{S})$ in which there are some slack variables $\widetilde{s}_r$ that are positive. In this case $\widetilde{X} \le X_p$ and $\sum_{j=1}^{n}\widetilde{\lambda}_j y_j - S \ge Y_p$ and at least one of the inequalities is strict. It means that $DMU_p$ is dominated and it's not strong-efficient. It contradicts the assumption.
In the second case; There is an optimal answer for a model such as $(\lambda, \widetilde{X}, \widetilde{S})$, where $\frac{C\widetilde{X}}{CX_p} < 1$ . This also contradicts the assumption.
Conversely, suppose $SCE_p = 1$. Using the"proof by contradiction method", let $DMU_p$ is not strong cost-efficient. two cases may occur; In the first case, there is an optimal solution for model ($\mathbf{2}$) such as $(\widetilde{\lambda}, \widetilde{X})$ that $CX < CX_p$. Obviously, $(\widetilde{\lambda}, \widetilde{X}, 0)$ is a possible solution for model ($\mathbf{6}$) with $SCE_p < 1$ and this contradicts the assumption. In the second case, $(\widetilde{\lambda}^*, X^*)$ is the optimal solution of model ($\mathbf{2}$) that $CX^* = CX_p$ i.e., $CE_p = 1$ and the optimal solution such as $(\widetilde{\lambda}, \widetilde{X})$ for model ($\mathbf{4}$) exists for which $1\widetilde{S} > 0$. Using these two optimal solutions, a possible solution such as $(\widetilde{\lambda}, X^*, \widetilde{S})$ can be constructed for model ($\mathbf{6}$). Clearly for this solution, we have $SCE_p < 1$ and this contradicts the assumption.

According to Model ($\mathbf{6}$), if we have $SCE_p < 1$ for $DMU_p$, it means that there is a virtual DMU such as $(\widetilde{X}, \widetilde{Y}) \in PPS$ that dominates $DMU_p$ and its cost is less than $DMU_p$. It means $C\widetilde{X} < CX_p$.The existence of such virtual DMU is easily verified by Model ($\mathbf{6}$). Conversely, if there is no DMU located on strong efficiency boundary that produces output the same as $DMU_p$ with lower cost, then it will have $SCE_p = 1$. Model ($\mathbf{6}$) is a fractional linear programming model that is transformed into the following linear programming model using the Charnes and Cooper transformations [2]:

$$SCE_p = \text{Min}\overline{\rho_p} = \frac{C\overline{X}}{CX_p}$$

s.t $\sum_{j=1}^{n}\overline{\lambda_j} x_{ij} = \overline{x_i},\ i = 1,\cdots,m,\ \overline{x_i} \le t x_{ip},\quad i = 1,\cdots,m,$

$$\sum_{j=1}^{n}\overline{\lambda_j} y_{rj} - \overline{s_r} = t y_{rp},\quad r = 1,\cdots,s, \qquad (\mathbf{6})$$

$t + \frac{1}{s}\sum_{r=1}^{s}\frac{\overline{s_r}}{y_{rp}} = 1,$
$\sum_{j=1}^{n}\overline{\lambda_j} = t,$
$\overline{\lambda_j} \ge 0,\ j = 1,\cdots,n,\ \overline{x_i} \ge 0,\ i = 1,\cdots,m,\ \overline{s_r} \ge 0,\ r = 1,\cdots,s,\ t \ge 0$

# 4 Numerical examples

**Example 2.** Consider fifteen DMUs with one input and one output according to Table 2. The technical efficiency of these units is evaluated by model (**1**) and is shown in the fourth column of Table 2. Also, the fifth column of this table includes the cost efficiency of these units, which is obtained by solving model (**2**). In this example, decision making units 1, 2, 3, and 4 use the same inputs. But their outputs are different and the first three units are dominated by $DMU_4$. This means that units 1, 2 and 3 are not cost-efficient. Using model (**2**), the cost efficiency of these four units is equal to one. Model (**6**) also gives the SCE, which is given in the sixth column of Table 2.

It is observed that in technical inefficient DMUs, $SCE < 1$ .Also, not all DMUs with necessarily have an $SCE = 1$. This is due to the presence of positive slack variables in the optimal solution of model (**2**).

Exmaple.2 data and results of solving models 1, 2 and 6

| DMU | Input $x$ | Output $y$ | Technical efficiency | Farell Cost Efficient | SCE |
|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 1 | 0.167 |
| 2 | 2 | 3 | 1 | 1 | 0.5 |
| 3 | 2 | 5 | 1 | 1 | 0.833 |
| 4 | 2 | 6 | 1 | 1 | 1 |
| 5 | 3 | 8 | 1 | 1 | 1 |
| 6 | 4 | 10 | 1 | 1 | 1 |
| 7 | 9 | 13 | 1 | 1 | 1 |
| 8 | 19 | 16 | 1 | 1 | 1 |
| 9 | 21 | 16 | 0.905 | 0.905 | 0.905 |
| 10 | 7 | 7 | 0.357 | 0.357 | 0.357 |
| 11 | 10 | 1 | 0.2 | 0.2 | 0.033 |
| 12 | 9 | 12 | 0.815 | 0.815 | 0.815 |
| 13 | 15 | 6 | 0.133 | 0.133 | 0.133 |
| 14 | 12 | 4 | 0.165 | 0.167 | 0.111 |
| 15 | 20 | 10 | 0.2 | 0.2 | 0.2 |

**Example 3.** Consider fifteen DMUs with two inputs and two outputs according to Table 3. The technical efficiency of these units is evaluated by model (**1**) and is shown in the sixth column of Table 3. Also, the seventh column of this table includes the cost efficiency of these units, which is obtained by solving model (**2**). Model (**6**) also gives the SCE, which is shown in the eighth column of Table 3. In this example, we also see that $DMU_5$ dominates $DMU_15$. These two units have the same technical efficiency and farell cost efficiency. But in these two units, only the SCE of unit number 5 is equal to one.

Exmaple.3 data and results of solving models 1, 2 and 6

| DMU | Input $x_1$ | Input $x_2$ | Output $y_1$ | Output $y_2$ | Technical efficiency | Farell Cost Efficient | SCE |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 12 | 1 | 10 | 1 | 0.684 | 0.173 |
| 2 | 1 | 10 | 3 | 10 | 1 | 0.813 | 0.614 |
| 3 | 1 | 8 | 4 | 10 | 1 | 1 | 1 |
| 4 | 2 | 6 | 6 | 9 | 1 | 1 | 1 |
| 5 | 4 | 3 | 8 | 6 | 1 | 1 | 1 |
| 6 | 9 | 1 | 9 | 3 | 1 | 1 | 1 |
| 7 | 12 | 1 | 9 | 2 | 1 | 0.778 | 0.513 |
| 8 | 3 | 7 | 7 | 1 | 1 | 0.63 | 0.1 |
| 9 | 6 | 8 | 3 | 2 | 0.5294 | 0.472 | 0.104 |
| 10 | 12 | 2 | 6 | 4 | 0.6765 | 0.567 | 0.297 |
| 11 | 2 | 12 | 5 | 5 | 0.75 | 0.425 | 0.286 |
| 12 | 10 | 10 | 1 | 7 | 0.3722 | 0.367 | 0.05 |
| 13 | 6.5 | 2 | 7 | 7.5 | 1 | 1 | 1 |
| 14 | 9.1 | 1 | 1 | 1 | 1 | 0.802 | 0.09 |
| 15 | 4 | 3 | 6 | 6 | 1 | 1 | 0.75 |

# 5 Conclusion

This paper presents a method for evaluating cost efficiency. Previous methods had a problem that by solving a model, the DMU may be introduced as cost-efficient, but not efficient. It is not true in the management point of view. Therefore, it seems logical that in addition to checking the cost-efficiency out, the performance of DMUs should also be considered. The model presented in this paper introduces the necessary condition for the cost efficiency of a DMU to be its Parato efficiency. To this purpose, the output constraint slack variables are being considered to evaluate cost-efficiency. If $SCE_p = 1$, it means that there is no DMU in the PPS whose outputs are are as much as that of $DMU_p$ And its production cost is less than that of $DMU_p$.

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# References

[1] R. D. Banker, A. Charnes and W. W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, Management Science, 30 (1984) 1078-1092.

[2] A. Charnes and W. W. Cooper, Programming with linear fractional functionals, Naval Research Logistic Quarterly, 9 (1962) 181-185.

[3] A. Charnes, W. W. Cooper and E. Rhodes, Measuring the efficiency of decision making units, European journal of operational research, 2 (1978) 429-444.

[4] W. W. Cooper, L. M. Seiford and K. Tone, Data envelopment analysis, A comprehensive text with models applications references, DEA solved software, Third printing, By Kluwer Academic Publishers (2002).

[5] R. Fare, S. Grosskopf and CAK. Lovell, The measurement of efficiency of production, Dordrecht MA, Kluwer, Boston (1985).

[6] M. J. Farrell, The measurement of productive efficiency, Journal of the Royal Statistical Society, Series A, 120 (1957) 253-290.

[7] K. Tone, A strange case of the cost and allocative efficiencies in DEA, Journal of the Operational Research Society, 53 (2002) 1225-1231.

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