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# Bayesian estimation of the parameters for two-parameter bathtub-shaped lifetime distribution based on ranked set sampling

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**Abstract**

Bayesian inference of the parameters for bathtub-shaped lifetime distribution based on simple random sampling (SRS) and ranked set sampling (RSS) are obtained. The Monte Carlo Markov chain is used, especially Metropolis-Hastings and Gibbs sampling. To compare different estimates, the Monte Carlo simulations are used. The results of simulation show that the estimators based on RSS are more efficient than based on SRS. Also, the length of highest posterior density (HPD) credible interval based on RSS is shorter than its SRS counterparts. Finally, a real data set has been analyzed for illustrative purposes.

## 1 Introduction

Several probability distributions are considered and studied for analyzing the bathtub-shaped failure rate data in literature. For example power exponential (Smith and Bain, 1975; Leemis, 1986), a reliability distribution with increasing, decreasing, and bathtub-shaped failure rate (Hjorth, 1980), Weibull exponentiated distribution (Mudholkar and Srivastava, 1993) and the two-parameter bathtub-shaped lifetime (TPBL) distribution (Chen, 2000; Wu et al., 2004, 2008; Ammar et al., 2012). The cumulative distribution function (cdf), the probability density function (pdf) and the failure rate function of the TPBL distribution with shaped and scale parameters ($\beta$, $\lambda$) are:

$$F(x) = 1 - e^{\lambda(1-e^{x^\beta})}, \quad x > 0, \ \ \lambda > 0, \ \ \beta > 0, \tag{1.1}$$

$$f(x) = \lambda\beta x^{\beta-1} e^{[x^\beta + \lambda(1-e^{x^\beta})]}, \quad x > 0, \ \ \lambda > 0, \ \ \beta > 0, \tag{1.2}$$

and

$$h(x) = \lambda\beta x^{\beta-1} e^{x^\beta}, \quad x > 0, \ \ \lambda > 0, \ \ \beta > 0. \tag{1.3}$$

The failure rate function $h(x)$ has a bathtub-shaped when $\beta < 1$ and increasing-shaped when $\beta \geq 1$, (chen (2000)).

Ranked set sampling (RSS) proposed by McIntyre (1952) and discussed in detail by Takahasi et al. (1968). This method of sampling have studied in different areas that actual measurements of the variable of study are time consuming or expensive, for example, in environmental (Patil et al., 1993), in reliability (Kvam et al., 1994) and in quality control (Muttlak et al., 2003). Comparing RSS with SRS in Bayesian context considered by Lavine (1999), Sadek and ALharbi(2014) and Sadek et al. (2015).

In order to obtain a sample of size $n$ by RSS method, we first take $n$ simple random samples of size $n$ from the population, and each sample is called a set. The items of each set are ranked by means of an auxiliary variable or visual judgment (without actual measurement). Then the $r^{th}$ observation of $r^{th}$ set is measured for $r = 1, \cdots, n$.

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This entire process denoted by $X_{(1)1}, \cdots, X_{(n)n}$ and is called a *cycle*. It may be repeated $k$ times to drive a RSS of size $m = nk$ from the underlying population where $n$ is the set size and k is the cycle size. Note that $X_{(r)r}$'s are independent and non-identically distributed (INID). The pdf and cdf of $X_{(r)r}$, denoted by $f_{r:n}(x)$ and $F_{r:n}(x)$, and are given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad x > 0,$$

and

$$F_{r:n}(x) = \sum_{i=r}^{n}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}, \quad x > 0,$$

respectively.

The organization of this paper is as follows: In Section 2, Bayesian estimation and HPD credible interval for the scale parameter $\lambda$ of TPBL distribution are considered based on SRS and RSS when $\beta$ is known using gamma and non-informative prior distributions. Also, Bayesian inference of the parameters $(\beta, \lambda)$ are provided in Section 3. Section 4 is devoted to simulation results. In Section 5 a real data set is analyzed for illustrative purposes. Finally, in Section 6 we conclude the paper by numerical examples.

# 2 Bayesian estimation of $\lambda$ when $\beta$ is known

Bayesian estimation of $\lambda$ is considered when $\beta$ is known based on SRS and RSS using gamma and non-informative prior distributions.

## 2.1 Bayes estimate of $\lambda$ based on SRS

Let $\mathbf{x} = x_1, x_2, \cdots, x_n$ be a SRS from TPBL$(\beta, \lambda)$ distribution, when $\beta$ is known. It is assumed that the parameter $\lambda$ follows a conjugate prior, with density:

$$\pi(\lambda) \propto \lambda^{a-1}e^{-b\lambda}, \qquad a \geq 0, \quad b \geq 0, \tag{2.1}$$

where a and b are known. In this case, the posterior density $\pi(\lambda|\mathbf{x})$ is given by

$$\pi(\lambda|\mathbf{x}) \propto \lambda^{n+a-1}e^{-\lambda\left\{b-\sum\limits_{i=1}^{n}(1-e^{x_i^{\beta}})\right\}}. \tag{2.2}$$

Therefore, $\lambda|\mathbf{x} \sim Gamma(n+a, b-\sum\limits_{i=1}^{n}(1-e^{x_i^{\beta}}))$. The posterior mean of $\lambda$ based on SRS is

$$\hat{\lambda}_{SRS} = E(\lambda|\mathbf{x}) = \frac{n+a}{b-\sum\limits_{i=1}^{n}(1-e^{x_i^{\beta}})}, \tag{2.3}$$

If $a = b = 0$, the prior (2.1) becomes the non-informative prior, which is given by $\pi(\lambda) \propto 1/\lambda, \quad \lambda > 0$. Then, the Bayesian estimation of $\lambda$ under non-informative prior is

$$\hat{\lambda}^{J}_{SRS} = -\frac{n}{\sum\limits_{i=1}^{n}(1-e^{x_i^{\beta}})}, \tag{2.4}$$

Furthermore, a $100(1-\alpha)\%$ highest posterior density (HPD) Bayesian credible interval of $\lambda$ can be constructed by using a numerical method as a subset $C \in \Lambda$ defined by $C = \{\lambda : \pi(\lambda|\mathbf{x}) \geq k\}$ where $k$ is the largest number such that

$$\int_{\lambda:\pi(\lambda|\mathbf{x})\geq k}\pi(\lambda|\mathbf{x})d\lambda = 1-\alpha. \tag{2.5}$$

## 2.2 Bayes estimate of $\lambda$ based on RSS

Let $Y_1, Y_2, \cdots, Y_n$ be a RSS from TPBL($\beta$, $\lambda$) distribution, the density of $Y_i$ is (see Arnold et al. (1992))

$$g(y_i|\lambda,\beta) = \frac{n!}{(i-1)!(n-i)!}[F(y_i|\lambda,\beta)]^{i-1}[1-F(y_i|\lambda,\beta)]^{n-i}f(y_i|\lambda,\beta)$$

$$= \sum_{k=0}^{i-1} i\binom{n}{i}\binom{i-1}{k}(-1)^k\lambda\beta y_j^{\beta-1}e^{y_j^\beta+\lambda(1-e^{y_j^\beta})(n+k-j+1)},$$

where $f$ and $F$ are pdf and cdf of the TPBL distribution. The joint density of $\mathbf{y} = (y_1, ..., y_n)$ is obtained as

$$g(\mathbf{y}|\lambda,\beta) = \prod_{j=1}^{n} g(y_j|\lambda,\beta) = \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}\left[\prod_{j=1}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}\right]$$

$$\times(\lambda\beta)^n(\prod_{j=1}^{n} y_j)^{\beta-1}e^{\sum_{j=1}^{n} y_j^\beta+\lambda\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})}. \tag{2.6}$$

If $\lambda$ follows prior distribution (2.1), then the posterior density of $\lambda$ is given by

$$\pi(\lambda|\mathbf{y},\beta) = \frac{\pi(\lambda)g(\mathbf{y}|\lambda,\beta)}{\int_0^\infty \pi(\lambda)g(\mathbf{y}|\lambda,\beta)d\lambda}$$

$$= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}]\lambda^{n+a-1}e^{-\lambda[b-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}]\Gamma(n+a)[b-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]^{-(n+a)}}.$$

Therefore, the Bayesian estimation of $\lambda$ based on RSS is

$$\hat{\lambda}_{RSS} = E(\lambda|\mathbf{y},\beta)$$

$$= \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}](n+a)[b-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]^{-(n+a+1)}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}][b-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]^{-(n+a)}}, \tag{2.7}$$

The Bayes estimates of $\lambda$ under non-informative prior are obtained as follows:

$$\hat{\lambda}^J_{RSS} = \frac{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}][-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]^{-(n+1)}}{\sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}][-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^\beta})]^{-n}}. \tag{2.8}$$

The form of the posterior pdf of $\lambda$ is so complex to solve the equation (2.5). We use MCMC technique to generate samples from it. Then, a $100(1-\alpha)\%$ HPD Bayesian credible interval of $\lambda$ can be constructed by applying the method proposed by Chen and Shao (1999) as

$$(\lambda_{[(\alpha/2)N]}, \quad \lambda_{[(1-\alpha/2)N]}), \tag{2.9}$$

where $\lambda_{[(\alpha/2)N]}$ and $\lambda_{[(1-\alpha/2)N]}$ are the $[(\alpha/2)N]$th smallest integer and the $[(1-\alpha/2)N]$th smallest integer of $\{\lambda_i, i = 1, \cdots, N\}$.

# 3 When $\beta$ and $\lambda$ are unknown

Bayesian estimation of both parameters of TPBL($\beta$, $\lambda$) distribution are discussed based on SRS and RSS using gamma and non-informative priors for each parameter.

## 3.1 Bayes estimates of $\beta$, $\lambda$ based on SRS

Assuming that $\beta$ and $\lambda$ have the joint prior density

(3.1) $$\pi(\beta,\lambda)=\pi(\beta).\pi(\lambda)\propto \beta^{a_1-1}e^{-b_1\beta}\lambda^{a_2-1}e^{-b_2\lambda},$$

using equations (1.2) and (3.1), the joint posterior density $\pi(\beta,\lambda|\mathbf{x})$ is given by

(3.2) $$\pi(\beta,\lambda|\mathbf{x})\propto \beta^{n+a_1-1}\lambda^{n+a_2-1}e^{-\beta(b_1-\sum_{i=1}^{n}\log x_i)+\sum_{i=1}^{n}x_i^{\beta}-\lambda(b_2-n+\sum_{i=1}^{n}e^{x_i^{\beta}})}.$$

Clearly, (3.2) may not reduced to a well known distribution. Therefore, we use Gibbs sampling method to generate values of $\beta$ and $\lambda$ from it. The full conditional density of $\lambda$ is obtained as

(3.3) $$(\lambda|\mathbf{x},\beta)\sim Gamma\left(n+a_2,b_2-n+\sum_{i=1}^{n}e^{x_i^{\beta}}\right),$$

and the conditional posterior density of $\beta$, satisfies

(3.4) $$\pi(\beta|\mathbf{x},\lambda)\propto \beta^{n+a_1-1}e^{-\beta(b_1-\sum_{i=1}^{n}\log x_i)+\sum_{i=1}^{n}x_i^{\beta}-\lambda\sum_{i=1}^{n}e^{x_i^{\beta}}}.$$

One can easily generates samples of $\lambda$ from (3.3). But since (3.4) is not a well known pdf, we use the Metropolis-Hastings (M-H) algorithm (Metropolis et al. (1953) and Hastings (2003)) to generate samples of $\beta$ with the half normal proposal density $q(\beta)\propto N(\beta^{(i-1)},\tau^2)I(\beta>0)$ , where the accept probability is

$$\rho=\min\left\{\frac{\pi(\beta^{(i)},\lambda^{(i)}|\mathbf{x})}{\pi(\beta^{(i-1)},\lambda^{(i)}|\mathbf{x})}\frac{q(\beta^{(i-1)}|\beta^{(i)},\mathbf{x})}{q(\beta^{(i)}|\beta^{(i-1)},\mathbf{x})},1\right\}.$$

The value of $\tau$ should be fixed so that the optimized value of $\rho$ is about 0.4. Therefore, the approximates of Bayesian estimation of $\beta$ and $\lambda$ based on SEL and LINEX are given by

(3.5) $$\hat{\beta}_{SEL}=\hat{E}(\beta|\mathbf{x})=\frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)},$$

(3.6) $$\hat{\beta}_{LINEX}=-\frac{1}{c}\ln[\hat{E}(e^{-c\beta}|\mathbf{x})]=-\frac{1}{c}\ln\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\beta^{(i)}}\right],$$

(3.7) $$\hat{\lambda}_{SEL}=\hat{E}(\lambda|\mathbf{x})=\frac{1}{N-M}\sum_{i=M+1}^{N}\lambda^{(i)},$$

(3.8) $$\hat{\lambda}_{LINEX}=-\frac{1}{c}\ln[\hat{E}(e^{-c\lambda}|\mathbf{x})]=-\frac{1}{c}\ln\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\lambda^{(i)}}\right],$$

where $M$ first iterations are taken as burn-in period. Note that if $a_1=a_2=b_1=b_2=0$, the prior (3.1) becomes non-informative.

### 3.2 Bayes estimates of $\beta$, $\lambda$ based on RSS

Using equations (2.6) and (3.1) the joint posterior distribution of $\beta$ and $\lambda$ based on RSS is obtained as

$$\pi(\beta,\lambda|\mathbf{y}) \propto \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}\prod_{j=0}^{n} j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}\beta^{n+a_1-1}$$

$$\times\,\lambda^{n+a_2-1}(\prod_{j=1}^{n} y_j)^{\beta-1} e^{-b_1\beta+\sum_{j=1}^{n} y_j^{\beta}-\lambda[b_2-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^{\beta}})]} \tag{3.9}$$

Similarly (3.2), since (3.9) does not have a well known distribution form, we use MCMC technique to generate samples from it. Note that

$$\pi(\beta|\mathbf{y},\lambda) \propto \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} c_{i_j}(j)]\beta^{n+a_1-1}(\prod_{j=1}^{n} y_j)^{\beta-1}$$

$$\times\exp[-b_1\beta+\sum_{j=1}^{n} y_j^{\beta}+\lambda\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^{\beta}})] \tag{3.10}$$

and

$$\pi(\lambda|\mathbf{y},\beta) \propto \sum_{i_1=0}^{0}\sum_{i_2=0}^{1}\cdots\sum_{i_n=0}^{n-1}[\prod_{j=0}^{n} c_{i_j}(j)]\lambda^{n+a_2-1}$$

$$\times\exp[-\lambda(b_2-\sum_{j=1}^{n}(n+i_j-j+1)(1-e^{y_j^{\beta}}))]. \tag{3.11}$$

where $c_{i_j}(j) = j\binom{n}{j}\binom{j-1}{i_j}(-1)^{i_j}$.

Since, no full conditional densities are known analytically, we rely again on M-H algorithm with two half normal proposals $q(\beta) \propto N(\beta^{(i-1)},\tau^2)I(\beta>0)$ and $q(\lambda) \propto N(\lambda^{(i-1)},\tau^2)I(\lambda>0)$.

By using equations (3.5) to (3.8) the approximations of Bayesian estimation of $\beta$ and $\lambda$ are derived.

## 4 Simulation Study

Monte Carlo simulations are used to approximate the Bayes estimators based on RSS and SRS for TPBL distribution.

### 4.1 Simulation for $\lambda$ when $\beta$ is known

Table 1 presents the mean squared error(MSE) and the bias of the estimators, for $\lambda = 0.5$ at different sizes of the sample [ $n = 4$, $n = 5$ and $n = 6$ ], when $\beta = 0.5$. We set $a = 2$ and $b = 3$ for Gamma prior. Note that the Bayesian estimates based on RSS have mean squared errors smaller than based on SRS. Also, in terms of bias it is observed that the bias via RSS is smaller than its SRS counterparts.

Furthermore, the mean squared errors of all estimates decrease when $n$ increases. The 95% HPD credible intervals of the Bayesian estimates for $\lambda$ are presented in Table 2. It is clear that, the HPD length based on RSS is shorter than its SRS counterparts.

Table 1: MSEs and biases of the Bayesian estimates based on SRS and RSS for $\lambda$ (when $\beta = 0.5$).

| $n$ | MSE | | | | bias | | | |
|---|---|---|---|---|---|---|---|---|
| | non-informative | | Gamma prior | | non-informative | | Gamma prior | |
| | SRS | RSS | SRS | RSS | SRS | RSS | SRS | RSS |
| 4 | 0.1784 | 0.0413 | 0.0596 | 0.0325 | 0.1577 | 0.0518 | 0.1233 | 0.0809 |
| 5 | 0.1502 | 0.0314 | 0.0565 | 0.0247 | 0.1566 | 0.0524 | 0.0651 | 0.0496 |
| 6 | 0.0894 | 0.0206 | 0.0473 | 0.0204 | 0.0756 | 0.0247 | 0.0925 | 0.0489 |

Table 2: The 95% HPD credible intervals of the Bayesian estimates based on SRS and RSS for $\lambda$ (when $\beta = 0.5$).

| $n$ | HPD | | | |
|---|---|---|---|---|
| | non-informative prior | | Gamma prior | |
| | SRS | RSS | SRS | RSS |
| 4 | (0.1233, 1.2810) | (0.2521, 0.9624) | (0.1835, 1.1260) | (0.2904, 0.9771) |
| 5 | (0.1614, 1.2261) | (0.2970, 0.8832) | (0.1904, 0.9892) | (0.3113, 0.8574) |
| 6 | (0.1697, 1.0388) | (0.3147, 0.7917) | (0.2316, 1.0601) | (0.3397, 0.8091) |

### 4.2 Simulation for $\beta$ and $\lambda$

The simulation study are based on 1000 Monte Carlo runs, and 5000 samples generated by using the M-H algorithm and discard the first M = 1000 values as burn-in period and $\tau = 0.5$.

The Bias and mean squared error of estimators, under SEL and LINEX loss function, for values $\beta = 0.5$ and $\lambda = 0.5$ at the sample sizes of $n = 5, 6$ and 7 are presented in Tables 3, when $a_1 = a_2 = 3$ and, $b_1 = b_2 = 2$ for Gamma prior. In Table 4, the bias and MSE of estimators are shown for non-informative prior.

From Tables 3 and 4, it is observed that the Bayesian estimates for $\beta$ and $\lambda$ based on RSS are in most cases less biased based on SRS. Also, the MSEs of estimators via RSS are obviously smaller than its SRS counterparts.

In context of loss function, from Tables 3 and 4, it can seen that MSE of the estimators under LINEX$_{\{C=1\}}$ are slightly smaller than the others. Furthermore, the MSE of all estimates decreases when $n$ increases.

The 95% HPD credible intervals of the Bayesian estimates based on SRS and RSS for $\beta$ and $\lambda$ under Gamma and non-informative priors are presented in Tables 5 and 6 respectively. It is observed that, the confidence length of HPDs based on RSS are shorter than its SRS counterparts.

We observe no much variation between estimators based on Gamma prior and non-informative prior especially based on RSS from the Tables 3-6.

The relative efficiency of the estimates based on RSS with respect to estimates based on SRS are calculated as follows:

$$efficiency = \frac{MSE(\hat{\theta}_{SRS})}{MSE(\hat{\theta}_{RSS})},$$

and their results are shown in Table 7. Clearly, estimators of $\beta$ and $\lambda$ based on RSS are more efficient than based on SRS, especially under LINEX$_{\{C=-1\}}$ loss function that the estimators via SRS have a relatively larger MSEs.

## 5 Real data analysis

In order to compare the performance of Bayes estimates of the parameters based on SRS and RSS, taking two samples of size 5 from the following data set that are display the number of 1000s of cycles to failure for electrical appliances in a life test originally reported by Lawless (2003)

0.014, 0.034, 0.059, 0.061, 0.069, 0.08, 0.123, 0.142, 0.165, 0.21, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.27, 1.275, 1.35,5 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.79, 3.857, 3.912, 4.1, 4.106, 4.116, 4.315, 4.51, 4.58, 5.267, 5.299, 5.583, 6.065, 9.701.

Table 3: MSEs and biases (in parenthesis) of the Bayesian estimates based on SRS and RSS for $\beta$ and $\lambda$ under Gamma prior.

| $n$ | SEL | | | | LINEX | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\beta$ | | $\lambda$ | | | $\beta$ | | $\lambda$ | |
| | SRS | RSS | SRS | RSS | C | SRS | RSS | SRS | RSS |
| 5 | 0.1892 (0.2897) | 0.0896 (0.1245) | 0.3023 (0.3094) | 0.0345 (0.0941) | 1 | 0.1577 (0.2617) | 0.0795 (0.1133) | 0.2214 (0.2542) | 0.0303 (0.0811) |
| | | | | | -1 | 0.2282 (0.3203) | 0.1011 (0.1361) | 0.3919 (-0.3105) | 0.0395 (0.1079) |
| 6 | 0.0751 (0.1345) | 0.0276 (0.0815) | 0.2540 (0.2906) | 0.0319 (0.0561) | 1 | 0.0640 (0.1188) | 0.0252 (0.0742) | 0.2115 (0.2381) | 0.0293 (0.0473) |
| | | | | | -1 | 0.0885 (0.1512) | 0.0302 (0.0888) | 0.3404 (-0.3267) | 0.0349 (0.0654) |
| 7 | 0.0689 (0.1613) | 0.0261 (0.0598) | 0.1257 (0.1443) | 0.0186 (0.0451) | 1 | 0.0607 (0.1479) | 0.0239 (0.0540) | 0.1029 (0.1162) | 0.0174 (0.0386) |
| | | | | | -1 | 0.0784 (0.1752) | 0.0295 (0.0656) | 0.1575 (0.1769) | 0.0199 (0.0519) |

Table 4: MSEs and biases (in parenthesis) of the Bayesian estimates based on SRS and RSS for $\beta$ and $\lambda$ under non-informative prior.

| $n$ | SEL | | | | LINEX | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\beta$ | | $\lambda$ | | | $\beta$ | | $\lambda$ | |
| | SRS | RSS | SRS | RSS | C | SRS | RSS | SRS | RSS |
| 5 | 0.1248 (0.1323) | 0.0435 (0.1005) | 0.1604 (0.1117) | 0.0265 (-0.0167) | 1 | 0.0946 (0.1052) | 0.0378 (0.0868) | 0.1026 (0.0595) | 0.0249 (-0.0281) |
| | | | | | -1 | 0.1709 (0.1631) | 0.0496 (0.1145) | 0.3718 (0.1967) | 0.0287 (-0.0043) |
| 6 | 0.0463 (0.0878) | 0.0301 (0.0214) | 0.0870 (0.0167) | 0.0162 (-0.0193) | 1 | 0.0400 (0.0696) | 0.0279 (0.0140) | 0.0701 (-0.0143) | 0.0156 (-0.0270) |
| | | | | | -1 | 0.0541 (0.1072) | 0.0324 (0.0289) | 0.3565 (-0.3370) | 0.0168 (-0.0112) |
| 7 | 0.0430 (0.0650) | 0.0203 (0.0796) | 0.0771 (0.0485) | 0.0142 (-0.0014) | 1 | 0.0357 (0.0511) | 0.0190 (0.0727) | 0.0620 (0.0191) | 0.0139 (-0.0074) |
| | | | | | -1 | 0.0522 (0.0797) | 0.0218 (0.0866) | 0.1135 (0.0837) | 0.0147 (0.0048) |

Table 5: The 95% HPD credible intervals of the Bayesian estimates based on SRS and RSS for $\beta$ and $\lambda$ under Gamma prior.

| $n$ | HPD | | | |
|---|---|---|---|---|
| | $\beta$ | | $\lambda$ | |
| | SRS | RSS | SRS | RSS |
| 5 | (0.3964, 1.2513) | (0.3691, 0.9048) | (0.3204, 1.5472) | (0.3231, 0.9430) |
| 6 | (0.3345, 0.9792) | (0.3657, 0.8084) | (0.3294, 1.4674) | (0.3309, 0.8494) |
| 7 | (0.3758, 0.9793) | (0.3648, 0.7666) | (0.2841, 1.1684) | (0.3469, 0.7914) |

Table 6: The 95% HPD credible intervals of the Bayesian estimates based on SRS and RSS for $\beta$ and $\lambda$ under non-informative prior.

| $n$ | HPD | | | |
|---|---|---|---|---|
| | $\beta$ | | $\lambda$ | |
| | SRS | RSS | SRS | RSS |
| 5 | (0.2613, 1.0939) | (0.3090, 0.9136) | (0.1735, 1.3267) | (0.2432, 0.8208) |
| 6 | (0.2657, 0.9776) | (0.3018, 0.7612) | (0.1644, 1.0932) | (0.2718, 0.7490) |
| 7 | (0.2791, 0.8940) | (0.3637, 0.8096) | (0.1961, 1.0927) | (0.3093, 0.7377) |

Table 7: Relative efficiency when $\beta$ and $\lambda$ are unknown.

| $n$ | non-informative | | | | | Gamma prior | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | SEL | | LINEX | | | SEL | | LINEX | | |
| | $\beta$ | $\lambda$ | C | $\beta$ | $\lambda$ | $\beta$ | $\lambda$ | C | $\beta$ | $\lambda$ |
| 5 | 2.8690 | 6.0528 | 1 | 2.5026 | 4.1205 | 2.1116 | 8.7623 | 1 | 1.9836 | 7.3069 |
| | | | -1 | 3.4456 | 12.9547 | | | -1 | 2.2572 | 9.9215 |
| 6 | 1.5382 | 5.3704 | 1 | 1.4337 | 4.4936 | 2.7210 | 7.9624 | 1 | 2.5397 | 7.2184 |
| | | | -1 | 1.6697 | 21.2202 | | | -1 | 2.9305 | 9.7536 |
| 7 | 2.1182 | 5.4296 | 1 | 1.8789 | 4.4604 | 2.6398 | 6.7581 | 1 | 2.5397 | 5.9138 |
| | | | -1 | 2.3945 | 7.7211 | | | -1 | 2.6576 | 7.9146 |

In the context of parameter estimation, Ammar et al.(2012) analyzed this data set based on the TPBL distribution. They showed that the TPBL model fits reasonably well to the data set. Therefore, we do not reproduce the results here for the sake of brevity. The drawn samples based on SRS and RSS are given in Table 8.

Now we compute the Bayesian estimations of $\beta$ and $\lambda$ based on SRS and RSS under SEL. Also, 95% HPD credible intervals of the Bayes estimates and the length of HPDs are constructed. All the results are based on the 5000 replications as reported in Table 9. It is obvious from this table that the length of HPD credible interval of $\beta$ based on RSS is smaller than the length of HPD credible interval of $\beta$ based on SRS. Also, the length of HPD of $\lambda$ via RSS is smaller than its SRS counterparts.

Furthermore, by using the Kolmogorov-Smirnov (K-S) and Anderson-Darling tests of goodness-of-fit we check the validity of the TPBL distribution in Table 10. It indicates that the TPBL distribution that is estimated by RSS, provides $p$-values larger than via SRS to the real data set. In addition, Figure 1 displays the histogram of the real data set and the fitted density functions, and Figure 2 shows the Q-Q Plot of the different sampling schemes. It is observed from these figures that the TPBL distribution that is estimated by RSS, provide a better fit than by SRS to the data.

# 6 Conclusions

In this work, we consider Bayesian inference of the parameters for TPBL distribution based on SRS and RSS. We obtain the Bayesian estimation of the scale parameter when the shaped parameter is known. Also, Bayes estimators for the both parameters are derived using MCMC method. Furthermore, we construct the HPD credible intervals for the parameters. By a simulation study difference between the estimation procedures are compared. The Bayes

Table 8: The number of 1000s of cycles to failure data based on SRS and RSS; $n = 5$.

| SRS | | | RSS | | |
|---|---|---|---|---|---|
| 1.893 | 0.123 | 0.556 | 1.355 | 0.210 | 0.969 |
| 5.267 | 2.993 | | 2.886 | 5.267 | |

Table 9: Bayesian estimates, 95% HPD credible intervals and HPD lengths of $\beta$ and $\lambda$.

| | $\hat{\beta}$ | HPD | HPD length | $\hat{\lambda}$ | HPD | HPD length |
|---|---|---|---|---|---|---|
| SRS | 0.4840 | (0.2139, 0.7730) | 0.5592 | 0.2869 | (0.0643, 0.6907) | 0.6263 |
| RSS | 0.5313 | (0.2839, 0.7601) | 0.4763 | 0.2542 | (0.1021, 0.4698) | 0.3677 |

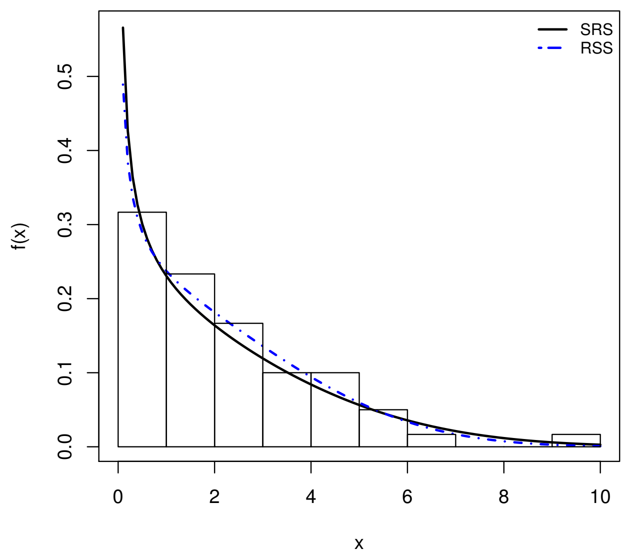


Figure 1: The histogram of real data set with the different fitted density functions.

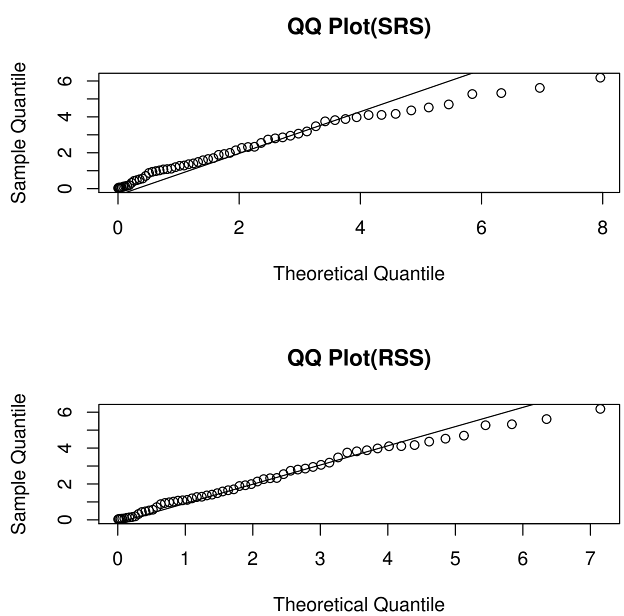


Figure 2: The QQ Plot of different sampling schemes.

Table 10: K-S and A-D tests for real data.

| | K-S test | | A-D test | |
|---|---|---|---|---|
| sampling scheme | Statistics $D$ | $p$-value | Statistics $An$ | $p$-value |
| SRS | 0.1031 | 0.5135 | 0.5973 | 0.6498 |
| RSS | 0.0673 | 0.9316 | 0.2610 | 0.9642 |

estimates based on RSS are perform better than the Bayes estimates based on SRS. In the context of HPD credible interval estimation, the HPDs based on RSS are shorter than the HPDs based on SRS.

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